

BILINEAR TRANSFORMATIONS IN HILBERT SPACE*

BY

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Introduction. A function of two variables $h = F(f, g)$, where h, f , and g are all elements of Hilbert Space may be termed a bilinear transformation if it is linear in f and linear in g . A more formal definition is given in §1. While a complete treatment of bilinear transformations would obviously require a very lengthy discussion, we wish to point out in this paper that many of the methods used in the study of linear transformations are applicable to them, with, of course, certain modifications. Many elementary notions can be extended and corresponding results obtained. For certain classes of bilinear transformations, there is even a "canonical resolution" (cf. §5, Theorem 7).

Bilinear transformations have appeared in the work of Kerner.† While the first Fréchet differential is a linear transformation, the second is bilinear, and it is this connection which was studied by Kerner. We shall show the relationship between bilinear transformations and rings of operators.‡

Mazur and Orlicz have pointed out the relationship between bilinear (and multilinear) transformations and polynomial transformations (cf. [5], p. 59). Polynomial transformations have also been studied by Banach (cf. [2]). We shall have occasion to use some of their results.

There is a very simple relationship between bilinear transformations and trilinear forms. For instance, if $F(f, g)$ is a bilinear transformation, then

$$\alpha(f, g, h) = (F(f, g), h),$$

(,) denoting the inner product, is linear in f and g , conjugate linear in h . For finite dimensional spaces, trilinear and multilinear forms have been discussed by Hitchcock and also by Oldenburger (cf. [3], [12], [13]). While a study of the infinite case demands more abstract methods and a decided shift in emphasis, nevertheless there is a certain similarity in the ideas involved in Hitchcock's paper and our discussion.

The results of §§1–5 can readily be extended to multilinear forms. In connection with this, it should be pointed out that in general we have a certain freedom in considering the nature of T (§4), in regard to the spaces on which it operates. For instance, if F is trilinear, we may consider T_1, T_2, T_3 defined by

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† Cf. [4]. Numerals in brackets refer to the bibliography at the end of this paper.

‡ Rings of operators are discussed in [9], [7], and [8].

$$\begin{aligned}(F(f, g, h), k) &= (T_1 f \otimes g \otimes h, k) = (T_2 f \otimes g, h \otimes k) \\ &= (T_3 f, g \otimes h \otimes k).\end{aligned}$$

Thus, if in the general case T_1 is a transformation from a k_1 -multiple to a k_2 -multiple space and T_2 is a transformation from a k_2 -multiple to a k_3 -multiple space, then $T_2 T_1$ is a transformation from a k_1 -multiple to a k_3 -multiple space and so can be regarded as determining a $(k_1 + k_3 - 1)$ -linear transformation. This last process corresponds to the notion of "composition" discussed by Oldenburger in [12].

In §1 we define a bilinear transformation and consider various possibilities for its domain. In §2 the notion of continuity and of a matrix for a bilinear transformation is discussed. Closure is discussed in §3 and its relationship with continuity. In §4 the "hypergraph" is discussed, and we show the connection between completely linear transformations $F(f, g)$ and linear transformations between $\mathfrak{H} \otimes \mathfrak{H}$ and \mathfrak{H} . In §5 the hyper-properties of completely linear transformations are discussed.

In §§6-12 we discuss the possibility of regarding Hilbert space as a hypercomplex number system. In §6 it is shown that this requires the introduction of a bilinear transformation $F(f, g)$ which is associative; that is, one for which $F(f, F(g, h)) = F(F(f, g), h)$. In §7 it is shown that if such an $F(f, g)$ is closed with respect to adjoints (Definition 7.2), then there is associated with it an algebraic ring of operators \mathbf{M} . If F also has certain continuity properties, then there is an element f_0 , such that the operation E_0 , defined by the equation $E_0 f = F(f_0, f)$, is a projection and is also the maximal idempotent for \mathbf{M} . The operation E_0 and the possible f_0 's are discussed further in §8. In §9 we discuss conditions which are sufficient for an \mathbf{M} and an f_0 to determine an associative bilinear transformation F .

In §10 examples of the foregoing are cited, and we also discuss the case in which an associative bilinear transformation is everywhere defined. In §11 for \mathbf{M} a ring of operators, we consider the relationship between F and \mathbf{M}' . In §12 we deal with the abelian cases of associative bilinear transformations.

Certain further examples are appended.

The second part of this paper represents also a development in a certain direction of a recent joint paper of J. von Neumann and the writer [8]. While the proofs given here were obtained independently, it is impossible to evaluate to what extent the general outline of the theory and the related notions have been influenced by previous discussions with Professor von Neumann. The author is also indebted to the referee for many suggestions in both parts of the present paper.

1. We introduce first the following definition:

DEFINITION 1.1. A function $F(f, g)$ of pairs of elements of Hilbert space is said to be a bilinear transformation if the following conditions hold:

- (a) The values of $F(f, g)$ are in Hilbert space.
- (b) If g is such that there is an f for which $F(f, g)$ is defined, then $R_g f = F(f, g)$ is a linear transformation on f .
- (c) If f is such that there is a g for which $F(f, g)$ is defined, then $T_f g = F(f, g)$ is a linear transformation on g .

The "graph" has yielded effective methods for the study of linear transformations (cf. [11]). For linear transformations, the method of procedure is to form* $\mathfrak{H} \oplus \mathfrak{H}$ and then to consider the set of pairs $\{f, Tf\}$ in this space. This set is called the graph. The statement that T is linear is equivalent to the statement that the graph is a linear manifold. The usefulness of the graph depends upon this fact.

Inasmuch as pairs of elements are involved, one might attempt to obtain a graph for bilinear transformations by forming $(\mathfrak{H} \oplus \mathfrak{H}) \oplus \mathfrak{H}$ and considering the elements of this space which are in the form $\{\{f, g\}, F(f, g)\}$. However the essential linearity property of $F(f, g)$ is the property that

$$F(f_1, g) + F(f_2, g) = F(f_1 + f_2, g), \quad F(f, g_1) + F(f, g_2) = F(f, g_1 + g_2).$$

We would therefore demand of the graph that

$$\{\{f_1, g\}, F(f_1, g)\} + \{\{f_2, g\}, F(f_2, g)\} = \{\{f_1 + f_2, g\}, F(f_1 + f_2, g)\}.$$

However the left-hand sum is

$$\{\{f_1, g\} + \{f_2, g\}, F(f_1, g) + F(f_2, g)\} = \{\{f_1 + f_2, 2g\}, F(f_1 + f_2, g)\}$$

by the usual rules for addition in the space $(\mathfrak{H} \oplus \mathfrak{H}) \oplus \mathfrak{H}$. In general, therefore, the desired equation does not hold.

This difficulty is easily traced to the fact that the linearity properties of $\{f, g\}$ are not the same as those of $F(f, g)$. However in the space $\mathfrak{H} \otimes \mathfrak{H}$ (cf. [7], loc. cit.) the expression $f \otimes g$ has precisely the same linearity properties as $F(f, g)$. As a consequence the elements of $(\mathfrak{H} \otimes \mathfrak{H}) \oplus \mathfrak{H}$ in the form $\{f \otimes g, F(f, g)\}$ are readily seen to represent the linearity properties of $F(f, g)$. It seems expedient therefore, to propose the following definition:

DEFINITION 1.2. The graph \mathfrak{G} of a bilinear transformation is that set of elements of $(\mathfrak{H} \otimes \mathfrak{H}) \oplus \mathfrak{H}$ in the form $\{f \otimes g, h\}$ for which $h = F(f, g)$. The domain of $F(f, g)$ is the set of elements $f \otimes g$ of $\mathfrak{H} \otimes \mathfrak{H}$ for which $F(f, g)$ is defined.

In Definition 1.1, two pairs $\{f, g\}$, $\{f^{(1)}, g^{(1)}\}$ are considered distinct un-

* The operations \oplus and \otimes as applied to Hilbert space, have been discussed in [14], Theorem 1.26, and [7], chap. 2, respectively.

less $f = f^{(1)}$, $g = g^{(1)}$. But $f \otimes g = f^{(1)} \otimes g^{(1)}$ does not imply $f = f^{(1)}$, $g = g^{(1)}$. However the situation is clarified by the following discussion. We begin with the statement that *if $f \otimes g = f^{(1)} \otimes g^{(1)}$ and if F is a bilinear transformation for which $F(f, g)$ and $F(f^{(1)}, g^{(1)})$ are defined, then $F(f, g) = F(f^{(1)}, g^{(1)})$.*

Two cases arise. If $f \otimes g = 0$, then $\|f \otimes g\| = \|f\| \cdot \|g\| = 0$, and either $f = 0$ or $g = 0$. In the first case $F(0, g) = R_0 0 = 0$. Similarly $F(f, 0) = 0$. Thus $f \otimes g = 0$ implies $F(f, g) = 0$. Therefore $f \otimes g = f^{(1)} \otimes g^{(1)} = 0$ implies that $F(f, g) = 0 = F(f^{(1)}, g^{(1)})$.

The case $f \otimes g \neq 0$ is shown by first proving that if $f \otimes g = f^{(1)} \otimes g^{(1)} \neq 0$, then $f^{(1)} = \lambda f$, $g^{(1)} = \mu g$ and $\lambda\mu = 1$. Under these circumstances both f and $g^{(1)}$ are not zero. It follows then that if we orthonormalize f , $f^{(1)}$ by the Gram-Schmidt process, we obtain either one, ϕ , or two, ϕ_1 and ϕ_2 , orthonormal elements. If the latter case could arise, then $f = a\phi_1$, $f^{(1)} = b_1\phi_1 + b_2\phi_2$ with $b_2 \neq 0$. Then $f \otimes g = f^{(1)} \otimes g^{(1)}$ may be written as

$$\phi_1 \otimes ag = \phi_1 \otimes b_1g^{(1)} + \phi_2 \otimes b_2g^{(1)}.$$

The argument in [7], §2.4, now implies that $b_2g^{(1)} = 0$. But since both b_2 and $g^{(1)}$ are not zero, this is impossible; hence only one orthonormal element ϕ can arise.

Thus $f = a\phi$, $f^{(1)} = b\phi$, and $ab \neq 0$. Hence $f^{(1)} = (b/a)f$. Also [7], §2.4, can now be used to show that $ag = bg^{(1)}$ or $g^{(1)} = (a/b)g$. Thus $f \otimes g = f^{(1)} \otimes g^{(1)} \neq 0$ implies $f^{(1)} = \lambda f$, $g^{(1)} = \mu g$, and $\lambda\mu = 1$. Consequently

$$F(f^{(1)}, g^{(1)}) = F(\lambda f, \mu g) = \lambda F(f, \mu g) = \lambda\mu F(f, g) = F(f, g).$$

These results show that while a pair $\{f \otimes g, h\}$ in the graph may represent more than one equation $h = F(f, g)$; nevertheless (except for $f \otimes g = 0$) each represented equation is a consequence of any other due to the nature of F .

Notice that it follows from Definition 1.1 that the set \mathfrak{N}_L of f 's for which $F(f, 0)$ is defined must be a linear manifold since $F(f, 0) = R_0 f$. Furthermore, it will contain the set \mathfrak{A}_L of all f 's for which $F(f, g)$ is defined for a nonzero g . We assume that \mathfrak{N}_L is precisely the linear manifold determined by \mathfrak{A}_L , unless an explicit extension is made. This, then, is the sense in which $0 = f \otimes 0$ is to be understood as in the domain of F .

We next discuss various possibilities for the domain of $F(f, g)$.

DEFINITION 1.3. *The domain of $F(f, g)$ is said to be dense if it is dense in the set of $f \otimes g$ of the space $\mathfrak{S} \otimes \mathfrak{S}$.*

DEFINITION 1.4. *The domain of $F(f, g)$ is said to be rectangular if, whenever it contains $f_1 \otimes g_1$ and $f_2 \otimes g_2$ both different from zero, it also contains $f_1 \otimes g_2$ and $f_2 \otimes g_1$.*

DEFINITION 1.5. *The domain of a bilinear transformation $F(f, g)$ is said to be completely linear if with $f_i \otimes g_i$, ($i=1, 2, \dots, n$), it also contains every element $f \otimes g$ such that $f \otimes g = \sum_{i=1}^n a_i f_i \otimes g_i$.*

LEMMA 1.1. *If the domain of a bilinear transformation $F(f, g)$ is rectangular, then the domain is completely linear.*

Proof. Suppose that $f_1 \otimes g_1, f_2 \otimes g_2, \dots, f_n \otimes g_n$ are in the domain of $F(f, g)$ and $f \otimes g = \sum_{i=1}^n a_i f_i \otimes g_i \neq 0$. Putting $a_0 = -1$, $f_0 = f$, $g_0 = g$, we may write $\sum_{i=0}^n a_i f_i \otimes g_i = 0$. Let us orthonormalize g_0, g_1, \dots, g_n by the Gram-Schmidt process, and let the result be $\phi_0, \phi_1, \dots, \phi_k$. Since $f_0 \otimes g_0 \neq 0$, $g_0 \neq 0$. Hence $C\phi_0 = g_0$, $C \neq 0$. Also for $i=1, 2, \dots, n$, $g_i = \sum_{j=0}^k b_{i,j} \phi_j$. Substituting we get

$$\begin{aligned} 0 &= \sum_{i=0}^n a_i f_i \otimes g_i = \sum_{i=0}^n \sum_{j=0}^k a_i b_{i,j} f_i \otimes \phi_j \\ &= \sum_{j=0}^k \left(\sum_{i=0}^n a_i b_{i,j} f_i \right) \otimes \phi_j \\ &= \left(-Cf + \sum_{i=1}^n a_i b_{i,0} f_i \right) \otimes \phi_0 + \sum_{j=1}^k \left(\sum_{i=0}^n a_i b_{i,j} f_i \right) \otimes \phi_j. \end{aligned}$$

This implies by [7], §2.4, that $-Cf + \sum_{i=1}^n a_i b_{i,0} f_i = 0$ or $f = \sum_{i=1}^n c_i f_i$. Now since the domain of $F(f, g)$ is rectangular, it must contain $f_i \otimes g_1$; hence by Definition 1.1 it must also contain $\sum_{i=1}^n c_i f_i \otimes g_1 = f \otimes g_1$.

A similar proof will show that $f_1 \otimes g$ is also in the domain of $F(f, g)$, and these two results imply, by Definition 1.2, that $f \otimes g$ is in the domain.

The converse of this lemma is not true (cf. Example 1 below).

DEFINITION 1.6. *The domain of $F(f, g)$ is said to be rectangularly dense if, for a dense set of g 's in \mathfrak{S} , R_g has a dense domain and if for a dense set of f 's in \mathfrak{S} , T_f has a dense domain (cf. Definition 1.1 above).*

It is easily seen that if the domain of $F(f, g)$ is rectangular and dense, it is rectangularly dense. The converse does not hold. The less restrictive condition is sufficient for some purposes.

DEFINITION 1.7. *The domain of $F(f, g)$ is said to be symmetric, if with $f \otimes g$ it contains $g \otimes f$.*

2. We next discuss certain elementary properties which F may have.

DEFINITION 2.1. *A bilinear transformation $F(f, g)$ is said to be continuous at a point $f_0 \otimes g_0$ of its domain if, given any $\epsilon > 0$, we can find a $\delta > 0$, such that when $f \otimes g$ is in the domain of F and $\|f - f_0\| < \delta$, $\|g - g_0\| < \delta$, then $\|F(f, g) - F(f_0, g_0)\| < \epsilon$. The transformation $F(f, g)$ will be said to be continuous if it is continuous at every point of its domain.*

This definition is, of course, the usual one for continuity in two variables. The notion of neighborhood implicit in it is equivalent to the neighborhood notion in $\mathfrak{S} \oplus \mathfrak{S}$. On the other hand, the additive properties of $F(f, g)$ are, as we shall see, best described in terms of $\mathfrak{S} \otimes \mathfrak{S}$. Despite this disharmony many results concerning the relation of additivity and continuity can be proved.

THEOREM 1. *If $F(f, g)$ has a rectangular domain and is continuous at one point, then $F(f, g)$ is continuous and there exists a constant C such that $\|F(f, g)\| \leq C\|f\| \cdot \|g\|$.*

It is a consequence of [5] (§11, p. 179) and Principle A (p. 59) that continuity at a single point implies continuity at every point. Thus $F(f, g)$ is continuous.

In particular, $F(f, g)$ is continuous at the origin. A familiar process used in [14] for the proof of Theorem 2.21, yields the existence of a C such that $\|F(f, g)\| \leq C\|f\| \cdot \|g\|$.

As with linear transformations, there exists a matrix theory for bilinear forms. The following theorem expresses this fact:

THEOREM 2. *Let $F(f, g)$ be bilinear. Then if ϕ_1, ϕ_2, \dots is a complete orthonormal set in \mathfrak{S} , there exists a set of bilinear complex-valued functions $\alpha_i(f, g)$ such that*

$$F(f, g) = \sum_{i=1}^{\infty} \alpha_i(f, g) \phi_i$$

for every $f \otimes g$ in the domain of F . If F is bounded, there exists for each i a bounded conjugate linear transformation T_i such that $\alpha_i(f, g) = (f, T_i g)$.

Proof. The first statement is shown by taking $\alpha_i(f, g) = (F(f, g), \phi_i)$ and applying Theorem 1.9 of [14]. If F has the bound C , then,

$$|\alpha_i(f, g)| = |(F(f, g), \phi_i)| \leq \|F(f, g)\| \cdot \|\phi_i\| \leq C\|f\| \cdot \|g\|.$$

A proof, very similar to that of Theorem 2.28 of [14] will now show the existence of T_i and its boundness under these circumstances.

3. We make the following definition:

DEFINITION 3.1. *If the graph \mathfrak{F} of $F(f, g)$ is closed, then $F(f, g)$ is said to be closed.*

Closure of the graph \mathfrak{F} is equivalent to the following statement concerning F : If $f_n \otimes g_n \rightarrow f \otimes g$ and $F(f_n, g_n) \rightarrow h$, then $f \otimes g$ is in the domain of F and $F(f, g) = h$.

We have the following relationship between closure and continuity for bilinear transformations:

THEOREM 3. *If $F(f, g)$ is defined for every pair $f \otimes g$ and is closed, then $F(f, g)$ is bounded.*

Proof. If we keep g fixed, $F(g, f) = T_g f$ is, under these hypotheses, a closed linear transformation with domain \mathfrak{F} . Hence by a well known result (cf. [1], chap. 3, Theorem 7, p. 41) T_g is bounded, with a bound C_g .

Similarly, if we keep f fixed, $F(g, f) = R_f g$ defines a bounded linear transformation R_f .

Now, there must be a neighborhood of 0 in \mathfrak{F} for which C_g is bounded. For otherwise, we can find a sequence g_n such that $g_n \rightarrow 0$ and $C_{g_n} \rightarrow \infty$ as $n \rightarrow \infty$. However for each f ,

$$\lim_{n \rightarrow \infty} T_{g_n} f = \lim_{n \rightarrow \infty} F(g_n, f) = \lim_{n \rightarrow \infty} R_f g_n = R_f 0 = 0.$$

Thus the T_{g_n} are a convergent sequence of bounded transformations; so by [1] (chap. 5, Theorem 5, p. 80) they are uniformly bounded. This contradicts the assumption that $C_{g_n} \rightarrow \infty$ as $n \rightarrow \infty$.

Thus there exist positive constants k and δ , such that $\|g\| \leq \delta$ implies $C_g \leq k$. Since $g = (\|g\|/\delta)g'$ for some g' with $\|g'\| = \delta$, we have

$$\|F(g, f)\| = (\|g\|/\delta)\|F(g', f)\| = (\|g\|/\delta)\|T_{g'} f\| \leq (k/\delta) \cdot \|g\| \cdot \|f\|.$$

This completes the proof of Theorem 3.

4. In the preceding sections, we dealt with the properties of $F(f, g)$ which are concerned with the graph \mathfrak{F} . This graph determines a linear manifold $\mathcal{L}(\mathfrak{F})$ in $(\mathfrak{F} \otimes \mathfrak{F}) \oplus \mathfrak{F}$. We now consider $\mathcal{L}(\mathfrak{F})$. First we make a definition as follows:

DEFINITION 4.1. *A bilinear transformation $F(f, g)$ will be said to be completely linear provided that the domain of $F(f, g)$ is completely linear and the relationship $f \otimes g = \sum_{i=1}^n a_i f_i \otimes g_i$ among the elements of the domain implies*

$$F(f, g) = \sum_{i=1}^n a_i F(f_i, g_i).$$

In the relationship $f \otimes g = \sum_{i=1}^n a_i f_i \otimes g_i$ we need only consider the case in which $f \otimes g \neq 0$. For if $f \otimes g = 0$ and at least one of the $a_i f_i \otimes g_i \neq 0$, then by applying the above definition to $n-1$ elements $f_i \otimes g_i$, we see that the equation on the values of F is still fulfilled. If all the $a_i f_i \otimes g_i = 0$, the same result obtains. Hence the above definition is equivalent to the corresponding one in which the condition $f \otimes g \neq 0$ is added.

It is important to note that a bilinear transformation is not necessarily completely linear, as we show by Example 2 below. The value of the notion of complete linearity lies in the following theorem:

THEOREM 4. *If a bilinear transformation $F(f, g)$ is completely linear, $l(\mathfrak{F})$ is the graph of a linear transformation T from $\mathfrak{S} \otimes \mathfrak{S}$ to \mathfrak{S} (cf. [6], Definition 1.2, p. 303). Also T is such that $f \otimes g$ is in the domain of T if and only if it is in the domain of $F(f, g)$; and when this occurs, $F(f, g) = Tf \otimes g$.*

Proof. Since $l(\mathfrak{F})$ is linear, it will be the graph of a transformation from $\mathfrak{S} \otimes \mathfrak{S}$ to \mathfrak{S} if and only if $\{0, h\} \in l(\mathfrak{F})$ implies $h=0$. Now if $\{k, h\}$ is in $l(\mathfrak{F})$, then there must exist a finite number of elements $f_i \otimes g_i$ in the domain of F and constants a_1, a_2, \dots, a_n , such that

$$k = \sum_{i=1}^n a_i f_i \otimes g_i, \quad h = \sum_{i=1}^n a_i F(f_i, g_i).$$

If $k=0$, then, as we have remarked after Definition 4.1 above, complete linearity implies $h=0$. This proves the first statement of the theorem.

The remaining statements are immediate consequences of the definition of $l(\mathfrak{F})$, Definitions 1.5 and 4.1 above, and Definition 1.2 of [6].

The converse of this is the following, the proof of which we omit:

THEOREM 5. *If T is a linear transformation from $\mathfrak{S} \otimes \mathfrak{S}$ to \mathfrak{S} , the equation $Tf \otimes g = F(f, g)$ determines a completely linear bilinear transformation $F(f, g)$, with domain the set of $f \otimes g$ which are in the domain of T .*

It is an immediate consequence of Definition 1.3 and the fact that the set of $f \otimes g$'s spans $\mathfrak{S} \otimes \mathfrak{S}$ that if the domain of F is dense and T exists, then the domain of T is dense.

THEOREM 6. *If the domain of a bilinear transformation $F(f, g)$ is rectangular, F is completely linear.*

Proof. We know from Lemma 1.1, that the domain of $F(f, g)$ is completely linear. Suppose now that $f_1 \otimes g_1, f_2 \otimes g_2, \dots, f_n \otimes g_n$ are in the domain of $F(f, g)$ and $f \otimes g = \sum_{i=1}^n a_i f_i \otimes g_i$. We must show that

$$F(f, g) = \sum_{i=1}^n a_i F(f_i, g_i).$$

Now by letting $a_0 = -1, f_0 = f, g_0 = g$, we see that this is a consequence of the statement that the relationship $\sum_{i=0}^n a_i f_i \otimes g_i = 0$, among elements in the domain of F , implies $\sum_{i=0}^n a_i F(f_i, g_i) = 0$.

We shall show this last statement inductively with respect to n . If $n=0$ and $a_0 f_0 \otimes g_0 = 0$, then a_0 or f_0 or g_0 is zero and hence $a_0 F(f_0, g_0) = 0$ by Definition 1.1. Now suppose it is true for $n-1$; we shall show it for n . We can suppose that $g_0 \neq 0$, since otherwise we have a situation equivalent to that of the case for $n-1$.

We can now proceed as in the proof of Lemma 1.1 to orthonormalize the g_0, g_1, \dots, g_n , obtaining $\phi_0, \phi_1, \dots, \phi_k$ with $g_i = \sum_{j=0}^k b_{i,j} \phi_j$, ($i=0, 1, \dots, n$). We then obtain, as there, that

$$0 = \sum_{i=0}^n a_i f_i \otimes g_i = \sum_{j=0}^k \left(\sum_{i=0}^n a_i b_{i,j} f_i \right) \otimes \phi_j;$$

and [7], §2.4, implies that

$$\sum_{i=0}^n a_i b_{i,j} f_i = 0, \quad j = 0, 1, \dots, k.$$

By Definition 1.4, $f_i \otimes g_j$ is in the domain of F for $i, j=0, 1, \dots, n$. Since the ϕ_j 's are linear combinations of the g_j 's, Definition 1.1 yields that $f_i \otimes \phi_j$ is in the domain of F for $i=0, 1, \dots, n$ and $j=0, 1, \dots, k$. Thus

$$\begin{aligned} \sum_{i=0}^n a_i F(f_i, g_i) &= \sum_{i=0}^n \sum_{j=0}^k a_i b_{i,j} F(f_i, \phi_j) \\ &= \sum_{j=0}^k F \left(\sum_{i=0}^n a_i b_{i,j} f_i, \phi_j \right) = \sum_{j=0}^k F(0, \phi_j) = 0. \end{aligned}$$

5. We deal in this section with completely linear $F(f, g)$.

DEFINITION 5.1. A completely linear $F(f, g)$ is said to be hypercontinuous if the T of Theorem 4 is continuous.

Since T is linear, the known results on linear transformations are immediately applicable here. For instance, we might define hypercontinuity at a point for $F(f, g)$ as continuity for T at the point $f \otimes g$ of $\mathfrak{S} \otimes \mathfrak{S}$. Then hypercontinuity at a point implies hypercontinuity. The correspondence of hypercontinuity and hyperboundness also results.

DEFINITION 5.2. A completely linear $F(f, g)$ is said to be hyperclosable if T possesses a closed extension $[T]$. Let $\bar{F}(f, g) = [T]f \otimes g$, as in Theorem 5.

The transformations F and \bar{F} are not, in general, equal, even if F is closed and possesses a rectangular and symmetric domain (cf. Example 3 below).

It should be pointed out that while it is obvious that the hyper-properties imply the corresponding simple properties of F , the converse is not true. Example 4 below is an example of an $F(f, g)$ which is bounded, has as domain all pairs $f \otimes g$, and yet is not even hyperclosable.

We next discuss hyperclosable transformations.

THEOREM 7. Let F be hyperclosable and $F = \bar{F}$. There exists a self-adjoint transformation H on $\mathfrak{S} \otimes \mathfrak{S}$ and a partially isometric transformation W from $\mathfrak{S} \otimes \mathfrak{S}$ to \mathfrak{S} such that the domain of F is exactly the set of $f \otimes g$ in the domain of H

and $F(f, g) = WHf \otimes g$. To each such H and W with the same zero manifold we can find an F with $F = \bar{F}$ such that $F(f, g) = WHf \otimes g$.

The canonical resolution of $[T]$ (cf. [6], Theorem 1.24, p. 312) yields this result immediately.

THEOREM 8. *If $F(f, g)$ is hyperclosable and $F = \bar{F}$, then the orthonormal set ϕ_1, ϕ_2, \dots of Theorem 2 can be chosen so that for each i there exists a conjugate linear transformation T_i of finite norm such that*

$$\alpha_i(f, g) = (f, T_i g).$$

Proof. Since $[T]$ is closed, we can by [6] (Theorem 6, p. 315) determine a set of mutually orthogonal closed linear manifolds \mathfrak{D}_i , ($i=0, \pm 1, \pm 2, \dots$), in $\mathfrak{H} \otimes \mathfrak{H}$ and a similar set \mathfrak{R}_i in \mathfrak{H} such that $[T]$ takes \mathfrak{D}_i into \mathfrak{R}_i . Let us choose in each \mathfrak{R}_i an orthonormal set $\phi_{i,1}, \phi_{i,2}, \dots$, complete in \mathfrak{R}_i .

From the definition of \mathfrak{D}_i and \mathfrak{R}_i in [6], Theorem VI, by means of [6], Theorem IV (we take $F' = WFW^*$) and [6], Theorem 1.24, we see that \mathfrak{R}_i is in the domain of $[T]^*$. Hence, for all \bar{f} in the domain of $[T]$,

$$[T]\bar{f} = \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} ([T]\bar{f}, \phi_{i,j}) \phi_{i,j} = \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} (\bar{f}, [T]^* \phi_{i,j}) \phi_{i,j}.$$

Now rearranging the $\phi_{i,j}$'s into a single sequence $\{\psi_k\}$ and letting $\bar{g}_k = [T]^* \phi_{i,j} = [T]^* \psi_k$, we obtain that for all \bar{f} in the domain of $[T]$

$$[T]\bar{f} = \sum_{i=1}^{\infty} (f, \bar{g}_i) \psi_i.$$

We next choose a complete orthonormal sequence $\{\phi_l\}$ in \mathfrak{H} . Then

$$\bar{g}_i = \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} a_{l,k}^{(i)} \phi_l \otimes \phi_k$$

with

$$\sum_{l=1}^{\infty} \sum_{k=1}^{\infty} |a_{l,k}^{(i)}|^2$$

finite. Let $\bar{f} = f \otimes g$ be in the domain of $[T]$. Since $F = \bar{F}$, it is also in the domain of T . Let $f = \sum_{i=1}^{\infty} x_i \phi_i$, $g = \sum_{j=1}^{\infty} y_j \phi_j$. Let T_i be the conjugate linear transformation such that

$$T_i g = \sum_{l=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{l,i}^{(i)} y_j \right) \phi_l.$$

Then T_i is of finite norm and

$$\begin{aligned}
 (f, T_i g) &= \sum_{l=1}^{\infty} x_l \left(\sum_{j=1}^{\infty} \bar{a}_{l,j}^{(i)} y_j \right) = \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} x_l y_j \bar{a}_{l,j}^{(i)} \\
 &= \left(\sum_{l=1}^{\infty} \sum_{j=1}^{\infty} x_l y_j \phi_l \otimes \phi_j, \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} a_{l,j}^{(i)} \phi_l \otimes \phi_j \right) \\
 &= (f \otimes g, \bar{g}_i) = \alpha_i(f, g).
 \end{aligned}$$

This proves the theorem.

Theorems 2 and 8 indicate that the relation between the set of hyperclosable and that of the bounded bilinear transformations is analogous to that between the linear transformations of finite norms and the bounded transformations.

6. An arbitrary vector space becomes a hypercomplex number system if a "rule of multiplication" $f \times g$ is given. Addition in the system is usually the vector sum $f + g$ of the original space. If, in particular, we consider Hilbert space, we are led to consider the ways in which a "multiplication rule" \times may be defined on the space.

Such an \times operation should be distributive; that is, we should have $(f_1 + f_2) \times g = f_1 \times g + f_2 \times g$ and $f \times (g_1 + g_2) = f \times g_1 + f \times g_2$. It should also be such that $af \times bg = ab(f \times g)$. So if we hold f fixed, we see that $T_f g = f \times g$ is a linear transformation on g . Similarly when g is fixed, $R_g f = f \times g$ is a linear transformation of f . Thus $F(f, g) = f \times g$ is a bilinear transformation in Hilbert space.

The associativity property, $f \times (g \times h) = (f \times g) \times h$, is also desired. For the corresponding bilinear transformation this would mean that $F(f, F(g, h)) = F(F(f, g), h)$. Since, however, it is too restrictive to demand that $F(f, g)$ be defined for every f and g , we define associativity for bilinear transformations as follows:

DEFINITION 6.1. *A bilinear transformation $F(f, g)$ is said to be associative if it satisfies the following conditions:*

- (a) *For every g , $F(f, g)$ is defined for every f in a linear set \mathfrak{A} .*
- (b) *$T_f g = F(f, g)$, ($f \in \mathfrak{A}$), is a bounded linear transformation.*
- (c) *If f_1 and f_2 are in \mathfrak{A} , $F(f_1, f_2)$ is in \mathfrak{A} .*
- (d) *For f_1 and $f_2 \in \mathfrak{A}$ and $g \in \mathfrak{S}$,*

$$F(f_1, F(f_2, g)) = F(F(f_1, f_2), g).$$

Thus if $f \times g$ has a domain of definition of the type given in Definition 6.1, it is an associative bilinear transformation. The problem of studying the ways in which Hilbert space can be regarded as a hypercomplex number system is, therefore, the analysis of associative bilinear transformations.

7. We shall show in this section a relationship between the study of as-

sociative bilinear transformations and that of rings of operators of a certain type (cf. [9]). It is customary to assume for a ring of operators \mathbf{M} that if A is in \mathbf{M} , A^* is also in \mathbf{M} . In order therefore to present a precise connection with the theory of rings of operators in its present form, we make the following definition:

DEFINITION 7.1. *An associative bilinear transformation $F(f, g)$ will be said to be closed with respect to adjoints if, whenever f is in \mathfrak{A} , there exists an $f^* \in \mathfrak{A}$ such that $T_f^* = (T_f)^*$.*

A preliminary connection is the following:

THEOREM 9. *The set \mathbf{M} of T_f 's associated with a given bilinear transformation $F(f, g)$ form an algebraic ring of operators† for which $T_f \cdot T_g = T_{F(f, g)}$ if and only if $F(f, g)$ is associative and closed with respect to adjoints.*

Suppose $F(f, g)$ is associative and closed with respect to adjoints. It follows from Definition 6.1 that each T_f is bounded. Also if $A = T_f$ and $B = T_g$ are in \mathbf{M} , then $\alpha A = \alpha T_f = T_{\alpha f}$ is in \mathbf{M} and $A + B = T_f + T_g = T_{f+g}$ is in \mathbf{M} . Since F is closed with respect to adjoints, $A^* = (T_f)^* = T_{f^*}$ is in \mathbf{M} . Finally, since f and g are in \mathfrak{A} ,

$$ABh = T_f \cdot T_g h = F(f, T_g h) = F(f, F(g, h)) = F(F(f, g), h) = T_{F(f, g)} h.$$

Thus $AB = T_{F(f, g)}$ and $AB \in \mathbf{M}$.

Conversely, if the T_f 's form an algebraic ring in which $T_f \cdot T_g = T_{F(f, g)}$, then $F(f, g)$ must be defined for a linear set of f 's since the T_f 's are a linear set and $T_f + T_g = T_{f+g}$, $\alpha T_f = T_{\alpha f}$. Since an algebraic ring consists of bounded operators, each T_f is bounded. Since \mathbf{M} is multiplicative, $T_f \cdot T_g = T_{F(f, g)}$ is in \mathbf{M} and $F(f, g)$ is in \mathfrak{A} . Also

$$F(f, F(g, h)) = T_f \cdot T_g h = T_{F(f, g)} h = F(F(f, g), h).$$

Thus F is associative. Furthermore, since $A = T_f \in \mathbf{M}$ implies $A^* \in \mathbf{M}$, $A^* = T_{f^*}$ for some f^* and F is therefore closed with respect to adjoints.

Thus the study of \times operations, subject to the restriction of being closed with respect to adjoints, may be referred to the study of algebraic rings of operators. At present such rings are also subjected in practice to certain further continuity restrictions. We will not need in §§7 and 8 of this paper the full restrictions usually imposed in order to obtain our results, and it is possible that the full restrictions are not needed even in the original theory of rings of operators.

† A set of bounded operators \mathbf{M} forms an algebraic ring if, whenever A and B are in \mathbf{M} and α is a complex number, αA , A^* , $A + B$, and $A \cdot B$ are in \mathbf{M} . This definition is given in [9], p. 383.

DEFINITION 7.2. *An associative bilinear transformation $F(f, g)$ will be said to be closed with respect to strongly convergent sequences if, whenever a sequence T_{f_n} converges strongly to a $T \in \mathcal{B}$, $T = T_f$ for some $f \in \mathcal{A}$.*

Closure with respect to strongly convergent sequences is sufficient for us to obtain the essential property of the set of T_f 's which we need in the following discussion. This property is the existence of a maximal idempotent. For an algebraic ring \mathcal{M} , the maximal idempotent has been defined in [9], when it exists, as that projection in \mathcal{M} such that for every $A \in \mathcal{M}$, $A = AE = EA$. It is unique.

THEOREM 10. *If an associative bilinear transformation $F(f, g)$ is closed with respect to adjoints and strongly convergent sequences, the set \mathcal{M} of T_f 's is an algebraic ring closed with respect to strongly convergent sequences. Also \mathcal{M} contains a maximal idempotent $E_0 = T_{f_0}$.*

The first statement follows easily from Theorem 9 and Definition 7.2. We must show that an algebraic ring \mathcal{M} , closed with respect to strongly convergent sequences contains a maximal idempotent.

Now if A is in \mathcal{M} , we can find a self-adjoint $H \in \mathcal{M}$ whose zeros are precisely those of A . For let $H = A^*A$. Then $Af = 0$ implies $Hf = 0$, and $Hf = 0$ implies $0 = (A^*Af, f) = \|Af\|^2$.

The proof given in [9] (II, §2, pp. 389–390) shows that if an algebraic ring \mathcal{M} is closed with respect to strongly convergent sequences and if H is in \mathcal{M} , then $E(0-)$ and $1 - E(0)$, in the resolution of the identity for H , are each in \mathcal{M} . (The complete hypothesis that \mathcal{M} is strongly closed is not used.) Thus $1 - E(0) + E(0-)$, the projection on the complement of the zeros of H , is in \mathcal{M} .

Combining the results of the two preceding paragraphs we see that for every $A \in \mathcal{M}$, we can find a projection $E \in \mathcal{M}$, such that the set of zeros of E is the set of zeros of A .

We can now show that \mathcal{M} contains a maximal projection E' ; that is, E' is such that for every $E \in \mathcal{M}$, $E'E = E$. Now there is a sequence $\{E_n\}$ strongly dense in the set of $E \in \mathcal{M}$ (cf. [9], I, §4, pp. 386–388). Since $E'X$ is continuous in the strong topology, if $E'E_n = E_n$ for every n , then $E'E = E$ for every $E \in \mathcal{M}$. Thus it will be sufficient to find an E' which majorizes the E_n .

Let $A_n = E_1 + E_2 + \cdots + E_n$. Now $A_nf = 0$ if and only if $E_if = 0$ for $i = 1, 2, \dots, n$, for

$$(A_nf, f) = \sum_{i=1}^n (E_if, f) = \sum_{i=1}^n \|E_if\|^2.$$

Since \mathcal{M} is linear, A_n is in \mathcal{M} . By a preceding result, we can find a projec-

tion E'_n , whose zeros are precisely those of A_n . Hence $E'_n E_i = E_i$ for $i=1, 2, \dots, n$.

We also have, from the definition, $E'_1 \leq E'_2 \leq \dots$. It follows that the E'_i converge strongly to an E' . Since M is closed with respect to strongly convergent sequences, E' is in M and furthermore $E' E'_n = E'_n$ for every n . Hence if n is greater than or equal to i , $E' E_i = E' E'_n E_i = E'_n E_i = E_i$. Thus E' majorizes E_i for every i , and, as we have remarked above, this is sufficient to yield that E' is a maximal projection.

But the maximal projection E' is also a maximal idempotent. For we have $E' E = E$ for every $E \in M$. Taking adjoints, we also have $E = E E'$. Hence $A = E' A = A E'$ if A is a projection in M . But under these circumstances, this equation must hold for linear combinations of projections too and also for their strong sequential limits. Thus the equation holds for all self-adjoint operators A in M . Finally since an arbitrary bounded A in M is a linear combination of two self-adjoint operators, it holds for every A in M .

Thus $E_0 = E'$ is a maximal idempotent for M . Since it is in M , there exists an $f_0 \in \mathfrak{A}$ such that $T_{f_0} = E_0$.

8. In this section, we continue the discussion of E_0 .

LEMMA 8.1. *Let $F(f, g)$ and E_0 be as above (Theorem 10). E_0 is the projection on the complement of those g 's such that $F(f, g) = 0$ for every $f \in \mathfrak{A}$.*

Proof. If g is such that $F(f, g) = 0$ for every $f \in \mathfrak{A}$, then $E_0 g = F(f_0, g) = 0$. If $g = (1 - E_0)g$, then

$$F(f, g) = T_f g = T_f(1 - E_0)g = (T_f - T_f E_0)g = 0 \cdot g = 0.$$

LEMMA 8.2. *If f is in \mathfrak{A} , $E_0 f$ is in \mathfrak{A} . If f is in \mathfrak{A} and $(1 - E_0)f = f$, then $T_f = 0$. If f is in \mathfrak{A} then $T_f = T_{E_0 f}$. If \mathfrak{M} is the closure of \mathfrak{A} , then the projection E on \mathfrak{M} commutes with E_0 .*

Proof. If f is an \mathfrak{A} , then $E_0 f = F(f_0, f)$ is in \mathfrak{A} by Definition 6.1. If f is in \mathfrak{A} and $(1 - E_0)f = f$, then

$$T_f = T_{f - E_0 f} = T_f - T_{E_0 f} = T_f - T_{F(f_0, f)} = T_f - E_0 T_f = 0.$$

Also if f is in \mathfrak{A} ,

$$T_f = T_{E_0 f} + T_{(1 - E_0)f} = T_{E_0 f}.$$

To show the last statement, we note that $f = E_0 f + (1 - E_0)f$ with $E_0 f \in \mathfrak{A}$; hence $(1 - E_0)f \in \mathfrak{A}$. This implies that the linear set $\mathfrak{A} = \mathfrak{N}' + \mathfrak{A}'$ where \mathfrak{A}' is included in the range of E_0 and \mathfrak{N}' in the orthogonal complement. The closure \mathfrak{M} is similar, and this implies the last statement of our lemma.

If \mathfrak{M} is as in Lemma 8.2, we can extend the definition of $F(f, g)$ so that $F(f, g)$ is defined for every $f \in \mathfrak{S} \ominus \mathfrak{M}$. For if $f = f_1 + f_2$, $f_1 \in \mathfrak{A}$, $f_2 \in \mathfrak{S} \ominus \mathfrak{M}$, we may define T_f as T_{f_1} . In the resulting extension, the properties of Definition 6.1, 7.1, and 7.2 are preserved (cf. the proof of Corollary 3 below) and, furthermore, \mathfrak{A} is dense.

DEFINITION 8.1. Let $F(f, g)$ be associative and closed with respect to adjoints and strongly convergent sequences. Let \mathfrak{N}_1 be the closure on the set of g 's in \mathfrak{A} such that $T_g = 0$. Let G_1 be the projection on \mathfrak{N}_1 . Then $F(f, g)$ will be said to be regular if, whenever f_0 is such that $T_{f_0} = E_0$ (cf. Theorem 10), $G_1 f_0$ is in \mathfrak{A} and $T_{G_1 f_0} = 0$.

Note that if the set of g 's for which $T_g = 0$ forms a closed linear manifold, then both conditions are fulfilled and $F(f, g)$ is regular. It will be shown in this section that regularity implies that there is an extension of F for which this is the case.

LEMMA 8.3. G_1 commutes with E_0 .

Proof. If g is such that $T_g = 0$, then $g = (1 - E_0)g + E_0 g$. By Lemma 8.2, $T_{(1-E_0)g} = 0$ and $T_{E_0 g} = T_g = 0$. Thus the set of g 's for which $T_g = 0$, is a linear manifold determined by a linear manifold in the range of E_0 and another in the complement of the range of E_0 . The closure \mathfrak{N}_1 has the same property; hence G_1 commutes with E_0 .

THEOREM 11. If $F(f, g)$ is regular, then f_0 (cf. Theorem 10) can be chosen in such a way that

- (a) $F(f_0, f_0) = f_0$,
- (b) f_0 is orthogonal to \mathfrak{N}_1 .

By Lemma 8.2, if $T_{f_0} = E_0$, then $T_{E_0 f_0} = E_0$. Thus we may choose f_0 so that $f_0 = E_0 f_0 = T_{f_0} f_0 = F(f_0, f_0)$.

Now if we let $f'_0 = f_0 - G_1 f_0$, where $E_0 f_0 = f_0$, then by the regularity of $F(f, g)$, f'_0 is in \mathfrak{A} and $T_{f'_0} = T_{f_0} - T_{G_1 f_0} = T_{f_0} = E_0$. Also by Lemma 8.3,

$$f'_0 = f_0 - G_1 f_0 = E_0 f_0 - G_1 E_0 f_0 = E_0 f_0 - E_0 G_1 f_0 = E_0 (f_0 - G_1 f_0) = E_0 f'_0,$$

and, as we have seen, this implies $f'_0 = F(f'_0, f'_0)$. Since $f'_0 = (1 - G_1)f_0$, f'_0 is orthogonal to \mathfrak{N}_1 .

COROLLARY 1. Let f_0 be as in Theorem 11, and let \mathfrak{A}_0 be the set of f 's in \mathfrak{A} in the form $F(f, f_0)$, $f \in \mathfrak{A}$. Then

- (a) \mathfrak{A}_0 is orthogonal to \mathfrak{N} ;
- (b) $f \in \mathfrak{A}_0$ and $T_f f_0 = F(f, f_0) = 0$ imply $f = 0$ and $T_f = 0$;
- (c) if g is in \mathfrak{A} , $g = g_0 + g_1$, where g_0 is in \mathfrak{A}_0 and g_1 is in \mathfrak{N}_1 , and $T_g = T_{g_0}$.

This resolution is unique, and $g_0 = F(g, f_0)$.

Proof of (a). Let g be such that $T_g = 0$. Then

$$(F(f, f_0), g) = (T_f f_0, g) = (f_0, (T_f)^* g) = (f_0, T_{f^*} g) = (f_0, F(f^*, g)).$$

Now $T_{F(f^*, g)} = T_{f^*} \cdot T_g = 0$. Hence $F(f^*, g)$ is in \mathfrak{N}_1 . Theorem 11, (b) now yields that

$$0 = (f_0, F(f^*, g)) = (F(f, f_0), g).$$

Thus $F(f, f_0)$ is orthogonal to \mathfrak{N}_1 . Hence \mathfrak{A}_0 is orthogonal to \mathfrak{N}_1 .

Proof of (b). Suppose $f \in \mathfrak{A}$ and $F(f, f_0) = 0$. We note that $T_{F(f, f_0)} = T_f \cdot T_{f_0} = T_f E_0 = T_f$. Thus $F(f, f_0) = 0$ implies $0 = T_0 = T_{F(f, f_0)} = T_f$. Thus $f \in \mathfrak{A}$ and $f \in \mathfrak{N}_1$. Hence by (a), $f = 0$.

Proof of (c). Suppose $g \in \mathfrak{A}$. Assume $g_0 = F(g, f_0) \in \mathfrak{A}$. Then $T_g = T_{F(g, f_0)}$; hence

$$T_{g_1} = T_{g-g_0} = T_g - T_{g_0} = 0.$$

Since \mathfrak{A}_0 and \mathfrak{N}_1 are orthogonal, the resolution is unique.

COROLLARY 2. *If f_1, f_2, \dots are each in \mathfrak{A}_0 (cf. Corollary 1), then*

(a) $F(f_i, f_0) = T_{f_i} f_0 = f_i$;

(b) *if the sequence T_{f_i} is strongly convergent with limit T , then the f_i converge to an $f \in \mathfrak{A}_0$, for which $T_f = T$.*

Proof of (a). $f_i - F(f_i, f_0)$ is in \mathfrak{A}_0 , since the latter is linear and in \mathfrak{N}_1 by Corollary 1, (c). Hence $f_i - F(f_i, f_0)$ is in $\mathfrak{A}_0 \cdot \mathfrak{N}_1$ and is zero by Corollary 1, (a).

Proof of (b). Since the T_{f_i} are convergent, $f_i = T_{f_i} f_0$ converges to $T f_0$. Since F is closed with respect to strongly convergent sequences (Definition 7.2), $T = T_g$ for a $g \in \mathfrak{A}$. Hence $f = T f_0 = T_g f_0 = F(g, f_0)$ is in \mathfrak{A}_0 , and by Corollary 1, (c) $T_f = T_g = T$.

Corollaries 1 and 2 give the relations for regular \times operations between the set of T_f 's and the f 's in \mathfrak{A} . Examples will be discussed in §10 below.

We discuss the significance of regularity in the following lemma:

LEMMA 8.4. *Suppose \mathfrak{A} is dense. If E_1 is the projection on \mathfrak{M}_1 , the closure of \mathfrak{A}_0 , then E_1 and $G_1 = 1 - E_1$ each commute with every T_f .*

Proof. If f is in \mathfrak{A}_0 , $f = F(f, f_0)$ and

$$T_g f = F(g, f) = F(g, F(f, f_0)) = F(F(g, f), f_0) \in \mathfrak{A}_0.$$

Thus if $f \in \mathfrak{A}_0$, then $T_g f$ is in \mathfrak{A}_0 . Since \mathfrak{M} is closed with respect to adjoints, $T_g^* f$ is also in \mathfrak{A}_0 . By continuity, therefore, if $f \in \mathfrak{M}$, then $T_g f \in \mathfrak{M}_1$ and $(T_g)^* f \in \mathfrak{M}_1$. Thus E_1 commutes with T_f (cf. [14], Theorem 4.25).

Since \mathfrak{A} is dense, it follows from Corollaries 1, (a) and (c) that \mathfrak{M}_1 and \mathfrak{N}_1 are orthogonal complements of each other. Hence $G_1 = 1 - E_1$ and, of course, commutes with T_f .

COROLLARY 3. If $F(f, g)$ is regular, F has a regular extension in which the set of g 's for which $T_g = 0$ form a closed linear manifold.

Suppose first that \mathfrak{A} is dense. Let g be in $\mathfrak{A} \cdot \mathfrak{N}$. By Corollary 1, (c) $g = g_0 + g_1$, $g_0 \in \mathfrak{A}_0$, $g_1 \in \mathfrak{N}_1$. Therefore $g_0 = g - g_1$ is also in \mathfrak{N}_1 and hence in $\mathfrak{A}_0 \cdot \mathfrak{N}_1$. By Corollary 1, (a), $g_0 = 0$. Hence $g = g_1$, $T_g = 0$.

Thus $\mathfrak{A} \cdot \mathfrak{N}_1$ consists of those g 's for which $T_g = 0$. Therefore we may extend \mathfrak{A} to the linear manifold determined by \mathfrak{N}_1 and \mathfrak{A} as follows. If $f = f_1 + f_2$, $f_1 \in \mathfrak{A}$, $f_2 \in \mathfrak{N}_1$, let $Tf = T_{f_1}$. This is an extension, since if T_f is already defined, f and f_1 are each in \mathfrak{A} , and thus $f_2 = f - f_1$ is in $\mathfrak{A} \cdot \mathfrak{N}_1$ and $T_{f_2} = 0$. A similar argument shows that T_f is unique for f in the extension of \mathfrak{A} .

We next show that extension $F'(f, g)$ is associative (Definition 6.1). Conditions (a) and (b) of that definition are immediately seen to be satisfied. To show (c), let f and g be elements of the extension of \mathfrak{A} . Then $f = f_1 + f_2$, $f_1 \in \mathfrak{A}$, $f_2 \in \mathfrak{N}_1$ and $g = g_1 + g_2$, $g_1 \in \mathfrak{A}$, $g_2 \in \mathfrak{N}_1$. Since $T_{f_2} = 0$, we have

$$F'(f_1 + f_2, g_1 + g_2) = F(f_1, g_1 + g_2) = F(f_1, g_1) + F(f_1, g_2) = F(f_1, g_1) + T_{f_2}g_2.$$

Since by Lemma 8.4, G_1 commutes with T_{f_1} , and since g_2 is in \mathfrak{N}_1 , $T_{f_1}g_2$ is in \mathfrak{N}_1 . Also $F(f_1, g_1)$ is in \mathfrak{A} and thus $F'(f_1 + f_2, g_1 + g_2)$ is in the extension of \mathfrak{A} . To show (d) we note that

$$\begin{aligned} F'(F'(f_1 + f_2, g_1 + g_2), h) &= F'(F(f_1, g_1) + T_{f_1}g_2, h) = F(F(f_1, g_1), h) \\ &= F(f_1, F(g_1, h)) = F'(f_1 + f_2, F'(g_1 + g_2, h)). \end{aligned}$$

Thus F' is associative.

Definitions 7.1 and 7.2, which are statements concerning the totality of T_f 's, unaffected in this extension, of course are satisfied by the extension.

In the case in which \mathfrak{A} is not dense, one first makes the extension given after Lemma 8.2. The E , which is the projection on \mathfrak{M} , the closure of \mathfrak{A} , commutes with every T_f . For $g \in \mathfrak{A}$ implies $T_fg = F(f, g) \in \mathfrak{A}$. An argument similar to that of Lemma 8.4 then yields that E commutes with T_f . The remainder of the justification for this extension is precisely similar to the argument given in the preceding paragraphs. One then makes the further extension given in this proof for the case in which \mathfrak{A} is dense.

It should also be remarked in connection with the regularity condition, that the restrictions of Definitions 6.1, 7.1, and 7.2 imply little with respect to the set of g 's for which $T_g = 0$. This set may even be dense. The regularity condition removes possibilities of this sort.

9. The previous sections have shown that the analyses of \times operations leads to an algebraic ring \mathbf{M} closed with respect to strongly convergent sequences. A further restriction on the nature of \mathbf{M} is implied in Corollary 1,

(b). This restriction is that there exists an f_0 such that $Tf_0=0$ implies $T=0$. If \mathbf{M} is also closed in the strong topology, a certain result ([9], Theorem 5, pp. 393–396) in the theory of rings of operators becomes available, and the nature of this restriction can be explored further.

THEOREM 12. *Let \mathbf{M} be a ring of operators (cf. [9], p. 388). Let E_0 be the maximal idempotent of \mathbf{M} (cf. the proof of Theorem 10). Then the necessary and sufficient condition that there should exist an f_0 such that $T \in \mathbf{M}$ and $Tf_0=0$ imply $T=0$ is that there should exist an $f' \in \mathfrak{S}$ such that $\mathfrak{M}_{E_0 f'}^{\mathbf{M}'}$ is the range of E_0 .*

Proof. Suppose that there is an f_0 such that $Tf_0=0$ and $T \in \mathbf{M}$ imply $T=0$. Since $T=TE_0$, this is equivalent to $TE_0f_0=0$ and $T \in \mathbf{M}$ imply $T=0$.

Consider $\mathfrak{M}_{E_0 f_0}^{\mathbf{M}'}$. Since $E_0 \in \mathbf{M}$ if $A \in \mathbf{M}'$, $AE_0f_0=E_0Af_0$. Thus $\mathfrak{M}_{E_0 f_0}^{\mathbf{M}'}$ is included in the range of E_0 . Furthermore if A is in \mathbf{M}' , $E_{E_0 f_0}^{\mathbf{M}'}$ commutes with A .[†] Hence $E_{E_0 f_0}^{\mathbf{M}'}$ is in \mathbf{M}'' . Since the range of $E_{E_0 f_0}^{\mathbf{M}'}$ is included in that of E_0 , $E_0E_{E_0 f_0}^{\mathbf{M}'}=E_{E_0 f_0}^{\mathbf{M}'}E_0=E_{E_0 f_0}^{\mathbf{M}'}$. The last two statements imply that $E_{E_0 f_0}^{\mathbf{M}'}$ is in \mathbf{M} by [9] (Theorem 5, pp. 393–396).

Thus $E_0 - E_{E_0 f_0}^{\mathbf{M}'}$ is also in \mathbf{M} . Since 1 is in \mathbf{M}' , $E_{E_0 f_0}^{\mathbf{M}'} \cdot E_0f_0 = E_0f_0$. Hence

$$(E_0 - E_{E_0 f_0}^{\mathbf{M}'})E_0f_0 = E_0^2f_0 - E_{E_0 f_0}^{\mathbf{M}'}E_0f_0 = E_0f_0 - E_0f_0 = 0.$$

Since $TE_0f_0=0$ and $T \in \mathbf{M}$ imply $T=0$, we have $E - E_{E_0 f_0}^{\mathbf{M}'}=0$ or $E_0 = E_{E_0 f_0}^{\mathbf{M}'}$.

Suppose, on the other hand, that there is an f' such that $\mathfrak{M}_{E_0 f'}^{\mathbf{M}'}$ is the range of E_0 . Suppose $Tf'=0$, $T \in \mathbf{M}$. Then, since $T=TE_0$, $TE_0f'=0$. Now if A is in \mathbf{M}' ,

$$TE_0AE_0f' = ATE_0^2f' = ATE_0f' = A \cdot 0 = 0.$$

Hence TE_0 is zero on the set of AE_0f , $A \in \mathbf{M}'$. Since this set is dense in the range of E_0 and TE_0 is bounded, this means that TE_0 is zero on the range of E_0 . Since $TE_0(1-E_0)=0$, it is zero on the orthogonal complement of this set also. Hence $TE_0=0$, and, since $T=TE_0$, $T=0$.

COROLLARY. *If \mathbf{M} is an algebraic ring closed with respect to strongly convergent series and such that there exists an f' for which $\mathfrak{M}_{E_0 f'}^{\mathbf{M}'} = \mathfrak{M}_0$, the range of E_0 , then $f_0 = E_0f'$ is such that $Tf_0=0$ and $T \in \mathbf{M}$ imply $T=0$.*

The last paragraph of the preceding proof shows this.

It may be noted here that there is another way of expressing the condition of Theorem 12 on \mathbf{M} . According to certain unpublished results of J. von

[†] Cf. [7], Definition 5.1.1, p. 143. However in this paper we drop the requirement of this definition that \mathbf{M} should be a prime. This will not affect our use of the known properties of $\mathfrak{M}_I^{\mathbf{M}}$.

[‡] Since A and A^* are both in \mathbf{M}' , they carry $\mathfrak{M}_{E_0 f_0}^{\mathbf{M}'}$ into part of itself. Paper [14], Theorem 4.25, now shows that $E_{E_0 f_0}^{\mathbf{M}'}$ commutes with A .

Neumann, an arbitrary ring \mathbf{M} may be expressed as the "sum" of rings \mathbf{M}_α which are factors on subsets \mathfrak{M}_α of \mathfrak{S} , there being a subset for each minimal projection of the center $\mathbf{M} \cdot \mathbf{M}'$ and "differential" subsets for the continuous spectrum of $\mathbf{M} \cdot \mathbf{M}'$. If \mathbf{M}_α is considered only on \mathfrak{M}_α , we may introduce \mathbf{M}'_α within \mathfrak{M}_α . The condition given above may be restated as follows. For every essential α , the normalized dimensionality of \mathbf{M}'_α must not be less than that of \mathbf{M}_α .

However, an algebraic ring \mathbf{M} which is subject to the restrictions given in the first paragraph of this section is the set of T_f 's for an \times operation of the type which we have considered in this paper.

DEFINITION 9.1. *Let \mathbf{M} be an algebraic ring of operators (cf. Theorem 9 above) which is closed with respect to strongly convergent series. Furthermore let \mathbf{M} be such that there exists an $f_0 \in \mathfrak{S}$, such that $T \in \mathbf{M}$ and $Tf_0 = 0$ imply $T = 0$. We may suppose $E_{f_0} = f_0$.*

Let \mathfrak{A}_0 consist of those elements of \mathfrak{S} in the form Tf_0 , $T \in \mathbf{M}$. Let \mathfrak{M}_1 be the closure of \mathfrak{A}_0 , E_1 the projection of \mathfrak{M}_1 . Let \mathfrak{N}_1 be the orthogonal complement of \mathfrak{M}_1 and G_1 the projection on \mathfrak{N}_1 .

We define $F_{\mathbf{M}, f_0}(f, g)$ (abbreviated to $F_{\mathbf{M}}(f, g)$) as follows. The transformation $F_{\mathbf{M}}(f, g)$ is defined whenever f is in the form $f_1 + f_2$, $f_1 \in \mathfrak{A}_0$, $f_2 \in \mathfrak{N}_1$ (and for all g). If, in these circumstances, $T \in \mathbf{M}$ is such that $f_1 = Tf_0$, then $F_{\mathbf{M}}(f, g) = Tg$.

THEOREM 13. *Under the assumptions of Definition 9.1, $F_{\mathbf{M}}(f, g)$ is single-valued. Furthermore $F_{\mathbf{M}}(f, g)$ is a bilinear transformation, for which the set of T_f 's is the \mathbf{M} of Definition 9.1.*

We show first that $F_{\mathbf{M}}(f, g)$ is single-valued. For a given f , there is at most one f_1 , since \mathfrak{A}_0 and \mathfrak{N}_1 are mutually orthogonal. Furthermore there is at most one $T \in \mathbf{M}$ such that $f_1 = Tf_0$. For if $f_1 = Tf_0$ and $f_1 = T'f_0$, T and $T' \in \mathbf{M}$, then $Tf_0 = T'f_0$ and $(T - T')f_0 = 0$. The assumptions of Definition 9.1 now imply that $T - T' = 0$ or $T = T'$.

We next show that $F_{\mathbf{M}}(f, g)$ is a bilinear transformation (Definition 1.1). The previous paragraph shows that for f fixed, the equation $T_f g = F(f, g) = Tg$ determines a linear transformation T_f . If g is fixed, $R_g f = F(f, g)$ also defines a linear transformation. For $R_g f$ is single-valued as we have seen above. Also, inasmuch as \mathbf{M} is linear, if $f_1 = Tf_0$, $g_1 = Sf_0$, T and $S \in \mathbf{M}$, then $af_1 = (aT)f_0$, $f_1 + g_1 = (T + S)f_0$. Hence

$$F(af, h) = F(a(f_1 + f_2), h) = F(af_1 + af_2, h) = aTh = aF(f_1 + f_2, h) = aF(f, h)$$

by Definition 9.1, and also

$$\begin{aligned} F(f + g, h) &= F(f_1 + g_1 + f_2 + g_2, h) = (T + S)h = Th + Sh \\ &= F(f, h) + F(g, h). \end{aligned}$$

Hence it follows that R_g is linear, and $F(f, g)$ is a bilinear transformation.

The first paragraph of this proof now shows that the set of T_f 's for $F_M(f, g)$ is the \mathcal{M} of Definition 9.1.

COROLLARY. F_M is regular, associative, and closed with respect to adjoints and strongly convergent sequences.

If $T = T_f, S = T_g$, then

$$\begin{aligned} F(f_1, g) &= F(f_1 + f_2, g_1 + g_2) = F(f_1, g_1 + g_2) = T_{f_1}(g_1 + g_2) \\ &= T_{f_1}g_1 + T_{f_1}g_2 = F(f_1, g_1) + T_{f_1}g_2. \end{aligned}$$

Now $E_1 = E_{f_0}^M$ is readily seen to commute with all the T_f by a familiar argument (cf. the footnote in the proof of Theorem 12). Hence $G_1 = 1 - E_1$ must also commute with every T_f , and since g_2 is in \mathfrak{N}_1 , $g' = T_{f_1}g_2$ is in \mathfrak{N}_1 . Hence $T_{g'} = 0$ and $T_{F_M(f, g)} = T_{F_M(f_1, g_1)} + T_{g'} = T_{F_M(f_1, g_1)}$. But $F(f_1, g_1) = T_{f_1}g_1 = Tg_1 = T \cdot Sf_0 \in \mathfrak{A}_0$; hence $T_{F_M(f_1, g_1)} = T \cdot S$. Thus $T_{F_M(f, g)} = T \cdot S = T_f \cdot T_g$.

Theorems 13 and 9 now imply that F is associative. Since \mathcal{M} is closed with respect to adjoints and strongly convergent sequences, Definitions 7.1 and 7.2 are satisfied. As we have remarked following Definition 8.1, the fact that the g 's for which $T_g = 0$ form a closed linear manifold insures regularity.

10. In this section we wish to discuss briefly known examples of \times operations.*

We first refer to a paper of J. von Neumann and the writer [8]. This memoir considers rings of operators \mathcal{M} called "factors in case II₁," which were discovered in a previous joint paper [7]. These rings, in the case $\alpha \geq 1$ (cf. [8], §1.1, p. 210), satisfy the assumption of Definition 9.1. For inasmuch as they are closed in the strong topology, they are closed with respect to strongly convergent sequences. Secondly, we may take for f_0 , the uniformly distributed g of [8] (Theorem II, p. 234). For the maximal idempotent E_0 of these rings is the identity 1, and the g of [8], Theorem II, shares with the f of [8], §1.1, the property that $\mathfrak{M}_f^{M'} = \mathfrak{M}_g^{M'} = \mathfrak{F}$. Theorem 12, above, now yields that such a g may be used as the f_0 of Definition 9.1.

If $\alpha = 1$, the $F_M(f, g)$ now resulting from the application of Definition 9.1 has an extension which is the \times operator of the algebra discussed in [8], chap. 4. The extension is obtained by taking the set of closed operators $Q_g(\mathcal{M})$, defined in [8], Definition 4.1.2, as the set of T_f 's. Since $Q_g(\mathcal{M})$ contains \mathcal{M} , this is an extension of the F_M which results from Definition 9.1 above.

If α is greater than 1, Definition 9.1 is still applicable. The \mathfrak{A}_0 will consist

* The author is indebted to the referee for the suggestion that the material of this section should be discussed.

of those f 's in the form Tg , $T \in \mathbf{M}$. Also \mathfrak{M}_1 , the closure of \mathfrak{A}_0 , is $\mathfrak{M}_g^{\mathbf{M}}$, and $E_1 = E_g^{\mathbf{M}}$. Since $D_{\mathbf{M}}(\mathfrak{F}) = \alpha > 1 = D_{\mathbf{M}}(E_1)$, we have a case in which E_1 is less than $E_0 = 1$.

The above examples illustrate the case in which \mathbf{M} is abstractly irreducible or, what is the same thing, the case in which \mathbf{M} has a minimal center, $\mathbf{M} \cdot \mathbf{M}' = \{\alpha 1\}$. The opposite extreme is the abelian case, in which $\mathbf{M} \cdot \mathbf{M}' = \mathbf{M}$. While we will give later a full discussion of the abelian case (cf. §12), we briefly point out certain simple examples here.

Suppose \mathfrak{F} is realized as \mathfrak{L}_2 , the set of square summable functions on the interval $0 \leq x \leq 1$ with

$$(f, g) = \int_0^1 f(x)\bar{g}(x)dx.$$

The set of operators \mathbf{M} defined by the equation $Tf = \phi(x)f(x)$, with $\phi(x)$ bounded, constitutes an abelian ring of operators, as one can readily verify. These operators also satisfy Definition 9.1, since they have the requisite closure property, and we may take f_0 as equal to the element $f(x) = 1$. For the resulting $F_{\mathbf{M}}$, we have $F_{\mathbf{M}}(\phi, g) = \phi(x)g(x)$.

An example which illustrates more completely the considerations of the previous sections is obtained by considering $\mathfrak{F} \oplus \mathfrak{L}_2 \oplus \mathfrak{L}_2$, the set of triples $\{f_1, f_2(x), f_3(x)\}$. Let \mathbf{M} consist of the operators defined by the equations

$$T\{f_1, f_2(x), f_3(x)\} = \{0, \phi(x)f_2(x), \phi(x)f_3(x)\}$$

in which $\phi(x)$ is bounded. Let $f_0 = \{0, 1, 0\}$. When Definition 9.1 is applied, E_0 is the projection on the set of elements in the form $\{0, f_2(x), f_3(x)\}$, and \mathfrak{M}_1 , the range of E_1 , is the set of elements in the form $\{0, f_2(x), 0\}$.

In the remainder of this section, we prove that if $F_{\mathbf{M}}(f, g)$ is everywhere defined and if \mathbf{M} is a ring of operators, then \mathbf{M} is finite dimensional; that is, \mathbf{M} has only a finite number of linearly independent elements. Inasmuch as it is necessary to appeal to the theory of rings of operators, our discussion must be limited to the case in which \mathbf{M} is closed in the strong topology.

But if $F_{\mathbf{M}}$ is everywhere defined, this restriction is not great. For instance, we can prove the following statement:

REMARK. *If $F_{\mathbf{M}}$ is everywhere defined and is closed (Definition 3.1), then \mathbf{M} is a ring of operators.*

We must show that \mathbf{M} is closed in the strong topology (cf. [9]).

First, we see from Theorem 3 of §3 above that $F_{\mathbf{M}}$ is continuous.

Secondly, let f be such that there exists a sequence f_n in \mathfrak{A} such that $T_{f_n}f_0 \rightarrow f$. Then

$$T_{f_n}f_0 = F(f_n, f_0) = F(f_n, F(f_0, f_0)) = F(F(f_n, f_0), f_0) = F(T_{f_n}f_0, f_0).$$

Since $F_M(f, g)$ is everywhere defined and continuous, $f = T_{f_0}f_0$. Then, since F_M is continuous, for every g ,

$$T_{f_n}g = F(f_n, g) = \lim_{n \rightarrow \infty} F(f_n, g) = \lim_{n \rightarrow \infty} T_{f_n}g.$$

Thirdly, let T be a limit point of M in the strong topology. We show that T is in M ; that is, $T = T_f$ for some f . For, let $f = T_{f_0}$, and let g be any element of \mathfrak{H} . Then since T is a strong limit of M , we can find a sequence of f_n 's such that $T_{f_n}f_0 \rightarrow T_{f_0}f_0 = f$ and $T_{f_n}g \rightarrow Tg$. Hence, by the above, $T_{f_n}g = \lim_{n \rightarrow \infty} T_{f_n}g = Tg$. Thus $T = T_f$, and M is closed in the strong topology.

LEMMA 10.1. *Let M be a ring of operators (cf. [9], p. 388). Furthermore let M be such that there exists an $f_0 \in \mathfrak{H}$, such that $T \in M$ and $Tf_0 = 0$ imply $T = 0$. Let A be an abelian ring in M . Let \mathfrak{B}_0 consist of those elements in the form Af_0 , $A \in A$. Let $[\mathfrak{B}_0]$ denote the closure of \mathfrak{B}_0 . Then $[\mathfrak{B}_0] \cdot \mathfrak{A}_0 = \mathfrak{B}_0$.*

Let f be in $\mathfrak{A}_0 \cdot [\mathfrak{B}_0]$. Then $f = T_{f_0}f$, $T \in M$. Since f is in $[\mathfrak{B}_0]$, we can find a sequence $\{A_n\}$, $A_n \in A$, such that $A_nf_0 \rightarrow f$ or $A_nf_0 \rightarrow T_{f_0}f$.

Let $\mathfrak{R} \subset \mathfrak{H}$ consist of those g 's such that $g = Bf_0$, $B \in M'$. Then $A_ng = A_nBf_0 = BA_nf_0 \rightarrow BT_{f_0}f = TBf_0 = Tg$. Thus $A_ng \rightarrow Tg$ for all $g \in \mathfrak{R}$.

Theorem 13 above states that \mathfrak{R} is dense in the range of E_0 . Since $A_nE_0 = A_n$ and $TE_0 = T$, $A_ng = Tg = 0$, for g in the complement of the range of E_0 . It follows that $A_ng \rightarrow Tg$ for a dense set of g 's.

However, it can also be shown that the A_n can be chosen so that they converge to a closed $A \eta A$.^{*} We present here merely an outline of the proof of this statement. The omitted details can easily be seen if use is made of the consideration of [14], chaps. 6 and 7. It is a consequence of [9] (III, §2, pp. 401-404) that there exist a resolution of the identity $E(\lambda)$ and bounded functions $\phi_n(\lambda)$ such that $A_n = \int_0^1 \phi_n(\lambda) dE(\lambda)$. Since $Af_0 = 0$ and $A \in A$ imply $A' = 0$, it follows that for $f \in \mathfrak{M}_{f_0}^A$, the equation $f = \int_0^1 \phi(\lambda) dE(\lambda) f_0$ determines ϕ essentially.[†] If

$$f = T_{f_0}f = \int_0^1 \phi(\lambda) dE(\lambda) f_0,$$

then, since $A_nf_0 \rightarrow f$, it follows that the $\phi_n(\lambda)$ approach ϕ essentially; that is,

$$\lim_{n \rightarrow \infty} \int_0^1 |\phi - \phi_n|^2 d\mu = 0.$$

^{*} The statement $A \eta A$, means that A has a dense domain, commutes with all unitary operators in A' , and is zero on the set on which all $A \in A$ is zero. This is a modification of [8], Definition 4.2.1.

[†] If $\mu(\lambda) = \|E(\lambda)f_0\|^2$, the word "essentially" refers to this μ . For instance, f determines ϕ except possibly on a set of μ -measure zero.

Now the A_n 's were any sequence in A such that $A_n f_0 \rightarrow f$. It is now clear that the A_n 's could be chosen in such a way that $|\phi_n(\lambda)|$ is an increasing sequence for each λ . These results are sufficient to show that if $A = \int_0^1 \phi(\lambda) dE(\lambda)$, then $A_n g \rightarrow Ag$ for every g in the domain of A and that the sequence A_n converges only on this domain.

Thus $Ag = Tg$ for all g in a dense linear set. But if T' is the contraction of T , whose domain is this set, T' has a unique closed extension, which is bounded. Thus $A = T$ and is bounded. Hence if $f = Tf_0$ is in $[\mathfrak{B}_0] \cdot \mathfrak{A}_0$, then $f = Af_0$, $A \in A$, or f is in \mathfrak{B}_0 . So $\mathfrak{B}_0 \supset [\mathfrak{B}_0] \cdot \mathfrak{A}_0$. But since obviously $\mathfrak{B}_0 \subset [\mathfrak{B}_0] \cdot \mathfrak{A}_0$, we have $\mathfrak{B}_0 = [\mathfrak{B}_0] \cdot \mathfrak{A}_0$.

LEMMA 10.2. *Let M be as in Lemma 10.1; then if M contains an infinite abelian ring A , $F_M(f, g)$ is not everywhere defined.*

Proof. As we pointed out in the proof of Lemma 10.1, each element of A is in the form $\int_0^1 \phi(\lambda) dE(\lambda)$ for a fixed resolution of the identity $E(\lambda)$. Furthermore $\phi(\lambda)$ is bounded. However, it is also easily seen that for every $\phi(\lambda)$ such that $\int_0^1 |\phi(\lambda)|^2 d\|E(\lambda)f_0\|^2$ exists, there exists an $f \in \mathfrak{F}$ such that $f = \int_0^1 \phi(\lambda) dE(\lambda)f_0$.

As we remarked above, if f is given, $\phi(\lambda)$ is essentially unique.

We next exhibit an unbounded ϕ which is such that

$$\int_0^1 |\phi(\lambda)|^2 d\|E(\lambda)f_0\|^2 < \infty.$$

Now inasmuch as A is infinite, we can divide the interval $(0, 1)$ into a denumerably infinite number of mutually exclusive subintervals I_n with adjoints a_n and b_n such that if $E(I_n) = E(b_n) - E(a_n)$, then $E(I_n) \neq 0$ and $E(I_n) \cdot E(I_m) = 0$ if $n \neq m$. Now $E(I_n)f_0 \neq 0$, for $E(I_n)f_0 = 0$ implies $E(I_n) = 0$, since $E(I_n)$ is in M . Also

$$\sum_{n=1}^{\infty} \|E(I_n)f_0\|^2 = \|f_0\|^2.$$

Let now $\alpha_n = \|E(I_n)f_0\|^2$. Then $\alpha_1 + \alpha_2 + \alpha_3 + \dots$ is a convergent sequence of positive terms. It is a well known result in the theory of infinite series, that we can find another convergent series of positive terms,

$$\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 \alpha_3 + \dots$$

such that $\lim_{n \rightarrow \infty} \lambda_n = \infty$.* Now for $x \in I_n$, let $\phi(x) = (\lambda_n)^{1/2}$. The function $\phi(x)$ is unbounded and

* Let R_n denote the remainder in the α series after the n th term. Let n_k denote that number such that for $n \geq n_k$, $2^k R_n \leq 1/2^k$. Now if n is such that $n_{k-1} < n \leq n_k$, let $\lambda_n = 2^{k-1}$. Then $\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \dots = R_0 - R_{n_1} + 2(R_{n_1} - R_{n_2}) + 2^2(R_{n_2} - R_{n_3}) + \dots \leq R_0 + 2R_{n_1} + 2^2R_{n_2} + \dots \leq R_0 + 1/2 + 1/2^2 + \dots = R_0 + 1$.

$$\int_0^1 |\phi(\lambda)|^2 d\|E(\lambda)f_0\|^2 = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots < \infty.$$

Now if $f = \int_0^1 \phi(\lambda) dE(\lambda)f_0$, then f is not in \mathfrak{B}_0 . For since f determines ϕ essentially, $f \in \mathfrak{B}_0$ implies that ϕ is bounded. But f is easily seen to be in $[\mathfrak{B}_0]$. For if we define ϕ_n as equal to $\phi(x)$, when $\phi(x)$ is less than or equal to n and equal to n otherwise, then $A_n = \int_0^1 \phi_n(\lambda) dE(\lambda)$ is in A and $f_n = A_n f_0$ is in \mathfrak{B}_0 . Furthermore $f = \lim_{n \rightarrow \infty} f_n$; so f is in $[\mathfrak{B}_0]$.

Now f is not in \mathfrak{A}_0 . For $f \in \mathfrak{A}_0$ implies $f \in [\mathfrak{B}_0]$. $\mathfrak{A}_0 = \mathfrak{B}_0$ (Lemma 10.1). Since f is in the closure of \mathfrak{A}_0 but not in \mathfrak{A}_0 , f is not in \mathfrak{A} .

Thus $F_M(f, g)$ is not defined.

The statement of the lemma is still valid even if we permit M to include closed unbounded operators which are limits of transformations in M on their domain but preserve, for the enlarged set, the property that $A \in M$ and $Af_0 = 0$ imply $A = 0$. For these the f in the proof of Lemma 10.2 is such that $T_f = \int_0^1 \phi(\lambda) dE(\lambda)$ since T_f is unique. But since T_f is unbounded, its domain is not the full space, and there are g 's for which $T_f g = F_M(f, g)$ is not defined.

LEMMA 10.3. *If M is infinite, it contains an infinite abelian ring.*

Suppose M contains only finite abelian rings. In particular, then the center $M \cdot M' = A$ is finite. This is easily seen to mean that there exists a finite number of mutually orthogonal projections, E_1, E_2, \dots, E_n , such that A is the set of transformations in the form $\sum_{i=1}^n a_i E_i$. We also know that E_0 is in $M \cdot M' = A$ and $E_0 E_i = E_i$ for $i = 1, 2, \dots, n$. Hence $E_0 = \sum_{i=1}^n E_i$.

Thus if A is in M ,

$$A = A E_0 = A \left(\sum_{i=1}^n E_i \right) = \sum_{i=1}^n A E_i = \sum_{i=1}^n E_i A E_i.$$

Let \mathfrak{M}_a denote the range of E_a , and let M_a denote the set of transformations in the form $A_a = E_a A E_a$, $A \in M$, considered only on \mathfrak{M}_a . The set M_a contains no elements A_a except those in the form $a \cdot 1$ which commute with every A_a . For if A_a is such, $E_a A_a E_a$ is in M , and also, as we see from the form of an arbitrary $A \in M$, it commutes with every $A \in M$. Hence $E_a A_a E_a$ is in $M \cdot M' = A$. This implies that $E_a A_a E_a$ is in the form $\sum_{i=1}^n a_i E_i$. Hence $E_a A_a E_a = a_a E_a$. Thus $A_a = a \cdot 1$ on \mathfrak{M}_a and $M_a \cdot M'_a = (a1)$.

Furthermore M_a does not contain an infinite abelian ring A_a . For if it did, the abelian ring A_a consisting of transformations in the form $E_a A_a E_a$, $A_a \in A$, would be an infinite abelian ring in M .

An analysis of rings for which $M \cdot M' = \{a \cdot 1\}$ is found in [7]. There are five types of such rings, cases I_n , I_∞ , II_1 , II_∞ , III_∞ (cf. [7], Theorem VIII, p. 172). It is a characteristic of cases II_1 , II_∞ , III_∞ , that they do not contain a

minimal projection (cf. [7], Definition 5.1.2). Hence they must contain an infinite abelian ring. Since I_∞ is isomorphic to the set of all operators on Hilbert space, it too contains infinite abelian rings. Hence since M_α is such that $M_\alpha \cdot M_{\alpha'} = (a \cdot 1)$ and does not contain an infinite abelian ring, M_α must be an I_n .

This, of course, means that M_α is a finite dimensional ring. For $A \in M$, we know that $A = \sum_{i=1}^n E_\alpha A_\alpha E_\alpha$, $A_\alpha \in M_\alpha$; so M itself is finite dimensional. We have thus shown that if M contains only finite abelian rings, it is finite.

THEOREM 14. *If $F(f, g)$ is a regular associative bilinear transformation such that the set of T_f 's forms a ring of operators (that is, is closed with respect to adjoints and also closed in the strong topology), then if $F(f, g)$ is everywhere defined, M is finite dimensional.*

It is a consequence of the discussion of §8, that if M is the set of T_f 's for $F(f, g)$, then $F(f, g) = F_M(f, g)$ (Definition 9.1). Since $F(f, g)$ is everywhere defined, we see from Lemma 10.2, that M does not contain an infinite abelian ring, and hence by Lemma 10.3 that it is finite dimensional.

The remark at the beginning of this discussion (preceding Lemma 10.1) now shows that the following statement is true:

COROLLARY. *If F is a regular associative bilinear transformation which is closed with respect to adjoints and closed (Definition 3.1) and if F is everywhere defined, then M , the set of T_f 's, is finite dimensional.*

11. If M is a ring of operators for which Definition 9.1 is applicable and if $F_M(f, g)$ is an associated bilinear transformation, then the R_α 's of F_M have certain properties which we discuss in this section. We let \mathfrak{A}_0 , \mathfrak{M}_1 , E_1 , \mathfrak{N}_1 , and G_1 be as in Definition 9.1. In conformity with §§7 and 8, we let E_0 be the maximal idempotent in M (cf. Theorem 10), \mathfrak{M}_0 the range of E_0 , and \mathfrak{A} the set of elements in the form $f_1 + f_2$, $f_1 \in \mathfrak{A}_0$, $f_2 \in \mathfrak{N}_1$.

There are certain relations in this situation which will be used without further comment. Thus \mathfrak{A} is dense by Definition 9.1. Lemma 8.4 yields $E_1 = 1 - G_1$. The second sentence of Lemma 8.2 implies $1 - E_0 \leq G_1$. These two relations yield further $E_1 \leq E_0$.

THEOREM 15. *Let M be a ring of operators (cf. [9], p. 388) such that Definition 9.1 is applicable. Let $F_M(f, g)$, E_1 , E_0 , and \mathfrak{A} be as in the preceding two paragraphs. Then*

(a) $R_\alpha f = F_M(f, g)$ is a linear transformation whose domain is \mathfrak{A} and $R_\alpha \eta M'$ (cf. [7], Definition 4.2.1, p. 141);

(b) the set of $[R_\alpha]$ for R_α bounded is exactly the set of $R \in M'$ for which $E_0 R = R = R E_1$.

Proof of (a). By the bilinearity of $F_M(f, g)$, R_g is a linear transformation. The domain of R_g is the set of f 's for which $F_M(f, g)$ is defined, and Definition 9.1 shows that this set is \mathfrak{A} . We must show $R_g \eta M'$. Now if A is in M' ,

$$A = A_1 + (1 - E_0)B(1 - E_0), \quad A_1 \in M,$$

and B can be quite arbitrary (cf. [9], Theorem 5, p. 393). Now since A_1 is in M , if $f_1 = A_1 f_0$, then for g arbitrary, $F_M(f_1, g) = A_1 g$. Also if h is in the domain of R_g , then

$$(1 - E_0)R_g h = (1 - E_0)F(h, g) = (1 - E_0)T_h g = (1 - E_0)E_0 T_h g = 0.$$

Hence $(1 - E_0) \cdot R_g = 0$. Furthermore, for h arbitrary, $(1 - E_0)h$ is in \mathfrak{N}_1 and hence in \mathfrak{A} , the domain of R_g . Also $T_{(1-E_0)h} = 0$. Thus

$$R_g(1 - E_0)h = F((1 - E_0)h, g) = T_{(1-E_0)h} g = 0;$$

so $R_g(1 - E_0) = 0$.

Thus for every h in the domain of R_g ,

$$\begin{aligned} AR_g h &= (A_1 + (1 - E_0)B(1 - E_0))R_g h = A_1 R_g h = A_1 F(h, g) \\ &= F(f_1, F(h, g)) = F(F(f_1, h), g) = R_g F(f_1, h) = R_g A_1 h \\ &= R_g(A_1 + (1 - E_0)B(1 - E_0))h = R_g A h. \end{aligned}$$

Hence $AR_g \subset R_g A$. Since this is true also for A^* , we see that $R_g \eta M'$.

Proof of (b). Since E_1 is the projection on the closure of \mathfrak{A}_0 , $1 - E_1$ is the projection on the zero's of R_g ; hence $R_g(1 - E_1) = 0$ or $R_g = R_g E_1$. From (a) above, we see that $R_g E_1 = R_g \eta M'$. Now R_g has domain dense. Hence if $R_g = R_g E_1$ is bounded, it has a closed extension $[R_g] \in M'$. Furthermore,

$$[R_g] = [R_g E_1] = [R_g E_1] \cdot E_1 = [R_g] E_1 = E_0 [R_g].$$

Let R be in M' and such that $RE_1 = R = E_0 R$. Let $g = Rf_0$. Then $g = E_0 Rf_0$ or g is in the range of E_0 . Hence $F_M(E_0 f_0, g) = E_0 g = g$. Consider R_g . For $f \in \mathfrak{A}$, we have

$$\begin{aligned} R_g f &= R_g T_f f_0 = R_g T_f E_0 f = T_f R_g E_0 f = T_f F_M(E_0 f, g) \\ &= T_f E_0 g = T_f g = T_f Rf_0 = RT_f f_0 = Rf, \end{aligned}$$

using the fact that R_g commutes with T_f and that $T_f = T_f E_0$. Hence R is an extension of R_g and R_g is bounded. Since the domain of R_g is dense, $R = [R_g]$. Thus the set of $[R_g]$ includes the set of elements of M' , for which $R = RE_1 = E_0 R$. The results of the previous paragraph now show that these two sets are equal.

COROLLARY 1. R_g is bounded for a dense linear set \mathfrak{D} of g 's.

Let us denote by \mathfrak{D}_0 , the set of g 's in the form Af_0 , $A \in \mathbf{M}'$. Since \mathbf{M} is a ring, Theorem 12 now states that \mathfrak{D}_0 is dense in the range of E_0 and $Af_0 = E_0Af_0$. But E_0 is in \mathbf{M}' , and Lemma 8.4 implies that E_1 is in \mathbf{M}' . Thus if A is in \mathbf{M}' , E_0AE_1 is in \mathbf{M}' . Also since f_0 is in \mathfrak{A}_0 , $E_1f_0 = f_0$. Hence $E_0AE_1f_0 = E_0Af_0 = Af_0$. So if g is in \mathfrak{D}_0 and $g = Af_0$, $A \in \mathbf{M}'$, we can suppose that $A = E_0A = AE_1$.

Theorem 15, (b) now implies that $A = [R_g]$; so R_g is bounded for $g \in \mathfrak{D}_0$. Furthermore, if g is in the range of $1 - E_0$ and $f \in \mathfrak{A}$, then

$$R_gf = F(f, g) = T_f g = T(1 - E_0)g = TE_0(1 - E_0)g = 0.$$

Since \mathfrak{A} is dense, $[R_g] = 0$. Thus R_g is bounded if $g = g_1 + g_2$, $g_1 \in \mathfrak{D}_0$, g_2 in the range of $1 - E_0$. Denote the set of such g 's by \mathfrak{D} . Since \mathfrak{D}_0 is dense in the range of E_0 , \mathfrak{D} is dense.

For $g \in \mathfrak{D}$, we may suppose that $F_{\mathbf{M}}(f, g)$ is defined for all f . For this extension $F_{\mathbf{M}'}(f, g)$ we have the following corollaries:

COROLLARY 2. *If \mathfrak{D} is as in the proof of Corollary 1, and f and g are in \mathfrak{D} , then $F_{\mathbf{M}'}(f, g)$ is in \mathfrak{D} .*

Let $f = f_1 + f_2$, $g = g_1 + g_2$, f_1 and $g_1 \in \mathfrak{D}_0$, f_2 and g_2 in the range of $1 - E_0$. By Lemma 8.2, $T_{f_2} = 0$. Since we also have $T_{f_1}g_2 = T_f(1 - E_0)g_2 = 0$,

$$F_{\mathbf{M}}(f, g) = F_{\mathbf{M}}(f, g_1 + g_2) = F_{\mathbf{M}}(f, g_1) = F_{\mathbf{M}}(f_1 + f_2, g_1) = F_{\mathbf{M}}(f_1, g_1).$$

Let $R_{f_1} = A$, $R_{g_1} = B$. Then

$$F_{\mathbf{M}}(f, g) = F_{\mathbf{M}}(f_1, g_1) = Bf_1 = BAf_0.$$

Since A and B are in \mathbf{M}' , BA is in \mathbf{M}' . Also

$$BA = (E_0B) \cdot A = E_0(BA), \quad BA = B(AE_1) = (BA)E_1.$$

Thus BA is an R_h by Theorem 15, (b). Furthermore, the proof of Theorem 15, (b) shows that since $F_{\mathbf{M}}(f, g) = BAf_0$, $BA = R_{F_{\mathbf{M}}(f, g)}$. Hence $F_{\mathbf{M}}(f, g)$ is in \mathfrak{D}_0 , which is included in \mathfrak{D} .

COROLLARY 3. *The set of R_g 's, $g \in \mathfrak{D}_0$, is closed in the strong topology.*

Proof. \mathbf{M}' is closed in the strong topology; hence the set of A 's of \mathbf{M}' for which $A = AE_1 = E_0A$ is also closed in the strong topology. Theorem 15, (b) now yields the corollary.

Suppose, for the moment, that Definition 6.1 had been defined with respect to the second variable of $F_{\mathbf{M}}(f, g)$ rather than the first. Corollary 1 is then the statement that conditions (a) and (b) are fulfilled by the extension of $F_{\mathbf{M}}$, and Corollary 2 is the same with respect to (c). Condition (d) is symmetric in the two variables. $F_{\mathbf{M}}$ thus satisfies Definition 6.1 in the new form.

Corollary 3 carries the topological closure property of the T_f 's over to the R_g 's. We next discuss closure with respect to adjoints for the R_g 's.

COROLLARY 4. *The set of R_g 's is such that for every $g \in \mathfrak{D}$ there exists a g^* such that $[R_g] = [R_g]^*$, if and only if $E_0 = E_1$.*

Suppose that for every $g \in \mathfrak{D}$ there exists a g^* such that $[R_g] = [R_g]^*$. Then, since for a closed bounded T we have $T^{**} = T$, every $[R_g]$ is the adjoint of a transformation $[R_g]$. Now

$$[R_g]^* = [R_g \cdot E_1]^* = ([R_g \cdot E_1] \cdot E_1)^* = E_1 [R_g \cdot E_1]^* = E_1 [R_g]^*.$$

Since $[R_g] = [R_g]^*$, we have $[R_g] = E_1 [R_g]$.

In the first paragraph of the proof of Corollary 1 of this section, it is shown that if, for an element f , $f = Af_0$ with A in \mathbf{M} , then A may be chosen so that $A = E_0 A = A E_1$. Hence by Theorem 15, (b), $A = [R_g]$ for some g . So if $f = Af_0$, $A \in \mathbf{M}'$, then $f = [R_g]f_0$. Hence

$$\begin{aligned} \mathfrak{M}_{E_0 f_0}^{\mathbf{M}'} &= [\{Af_0, A \in \mathbf{M}'\}] = [\{[R_g]f_0, g \in \mathfrak{D}\}] = [\{E_1 [R_g]f_0, g \in \mathfrak{D}\}] \\ &= E_1 [\{[R_g]f_0; g \in \mathfrak{D}\}] = E_1 \mathfrak{M}_{E_0 f_0}^{\mathbf{M}'} \end{aligned}$$

Theorem 12 states that $\mathfrak{M}_{E_0 f_0}^{\mathbf{M}'}$ is the range of E_0 . Thus $E_1 \mathfrak{M}_{E_0 f_0}^{\mathbf{M}'} = \mathfrak{M}_{E_0 f_0}^{\mathbf{M}'}$ implies $E_1 \geq E_0$. Since we also have $E_1 \leq E_0$, we have $E_1 = E_0$. Thus if for every $g \in \mathfrak{D}$ there exists a $g^* \in \mathfrak{D}$ such that $[R_g] = [R_g]^*$, then $E_1 = E_0$.

The converse of this result is given immediately by Theorem 15, (b).

The results of this section show that for regular associative bilinear transformations, $F_M(f, g)$, the properties of the two variables are the same if and only if $E_1 = E_0$. However if $E_1 = E_0$ this symmetry extends even further since the ring consisting of transformations in the form $E_1 A E_1$, $A \in \mathbf{M}'$, is related to the second variable as \mathbf{M} is to the first in Definition 9.1.

12. In the special case of a regular associative bilinear transformation in which the T_f commute, that is, $F(f, g) = F(g, f)$, f and $g \in \mathfrak{A}_0$, closure for strongly convergent sequences is equivalent to strong closure (cf. [9], III, §1, p. 398). The known analyses of self-adjoint operators and abelian rings then permit us to obtain more specific results.

THEOREM 16. *Let F be a regular associative bilinear transformation closed with respect to strongly convergent sequences and adjoints. Furthermore, let $F(f, g) = F(g, f)$ for f and $g \in \mathfrak{A}_0$ (cf. Corollary 1 of Theorem 11). Let E_0 , E_1 , and f_0 be as in §§7 and 8. Then*

(a) \mathbf{M} is generated by a self-adjoint transformation

$$H = \int_0^c \lambda dE(\lambda);$$

(b) if $\mu(\lambda) = \|E(\lambda)f_0\|^2$, there exists a sequence, finite or infinite, of μ -measurable sets, S_1, S_2, S_3, \dots , $S_1 \supseteq S_2 \supseteq S_3 \supseteq \dots$, such that \mathfrak{S} may be realized as $\mathfrak{S}_0 \oplus \mathfrak{S}_1 \oplus \mathfrak{S}_2 \oplus \dots$ (cf. [14], Theorems 1.26 and 1.27), where \mathfrak{S}_0 corresponds to the elements in the range of $1 - E_0$ and \mathfrak{S}_i is the space of μ -summable squared functions $\psi_i(\lambda)$ defined on S_i ;

(c) E_1 is the projection on the set of elements in the form $\{0, \phi_1(\lambda), 0, \dots\}$, \mathfrak{A}_0 the subset of these for which $\phi_1(\lambda)$ is essentially bounded, f_0 that one of these elements for which $\phi_1(\lambda) = 1$;

(d) the relations

$$F(\{0, \phi_1(\lambda), 0, \dots\}, \{g_1, \psi_1(\lambda), \psi_2(\lambda), \dots\}) = \{0, \phi_1(\lambda)\psi_1(\lambda), \phi_1(\lambda)\psi_2(\lambda), \dots\},$$

and

$$T_{\{0, \phi_1(\lambda), \dots\}} = \int_0^1 \phi_1(\lambda) dE(\lambda)$$

are satisfied;

(e) if $f = \{0, \phi_1(\lambda), 0, \dots\}$ is in \mathfrak{A}_0 , then $f^* = \{0, \bar{\eta}_1(\lambda), 0, \dots\}$.

Proof. M is, as we have seen above, an abelian ring, and this (cf. [9], III, §2, pp. 401–404) implies (a). We now apply the analysis of [14] (chap. 7, §2) to H considered only on the range of E_0 . We do not, however, distinguish between the point and continuous spectrum, the point spectrum representing merely discontinuities of the $\rho_i(\lambda)$. Since for $E \in M$, $Ef_0 = 0$ implies $E = 0$, we may take $\rho_1(\lambda) = \mu(\lambda)$. We obtain a sequence of functions of bounded variation $\rho_1 \prec \rho_2 \prec \rho_3 \prec \dots$, such that we can express the range of E_0 in the form $\mathfrak{S}_1 \oplus \mathfrak{S}_2 \oplus \mathfrak{S}_3 \oplus \dots$ where \mathfrak{S}_i is the space whose inner product is

$$\int_0^c \phi(\lambda) \bar{\theta}(\lambda) d\rho_i(\lambda) = \int_0^c \phi(\lambda) \bar{\theta}(\lambda) \frac{d\rho_i}{d\rho_1} d\rho_1.$$

Now if S_i is the set on which $d\rho_i/d\rho_1 \neq 0$ and if, to the element $\phi(\lambda)$ in this realization, we make $\phi' = \phi(\lambda)(d\rho_i/d\rho_1)^{1/2}$ correspond, we see that \mathfrak{S}_i may also be realized as the space whose inner product is $\int_{S_i} \phi'(\lambda) \bar{\theta}'(\lambda) d\rho_1$. We have essentially, except for the sets of μ -measure zero, $S_1 \supset S_2 \supset S_3 \supset \dots$, and this completes the proof of (b).

The above process has already identified \mathfrak{S}_1 with \mathfrak{M}_1 , and the remaining statements are immediate consequences of the operational calculus (cf. [14], chap. 6, or [10]).

Examples. We present here five examples.

EXAMPLE 1. We give first an example of a bilinear transformation whose domain is completely linear but not rectangular. Let $\{\phi_i\}$ be an orthonormal set containing at least two elements. Let $F(k\phi_i, l\phi_i) = kl\phi_i$ for each i . As to

when $F(f, 0)$ and $F(0, g)$ are defined, consult the remark following Definition 1.2. Then $F(f, g)$ is defined and not zero only if $g = l\phi_i$ for some l and i and then only for f 's in the form $k\phi_i$. For each such g it is obviously linear. A similar statement holds for f .

Now the domain of F is completely linear. For let $f \otimes g$ be such that $f \otimes g = \sum_{i=1}^n \lambda_i \phi_{n_i} \otimes \phi_{n_i}$. As in §1, Lemma 1.1 above, we see that $f = \sum_{i=1}^n c_i \phi_{n_i}$, $g = \sum_{i=1}^n d_i \phi_{n_i}$. Hence

$$\begin{aligned} \sum_{i=1}^n \lambda_i \phi_{n_i} \otimes \phi_{n_i} &= f \otimes g = \left(\sum_{i=1}^n c_i \phi_{n_i} \right) \otimes \left(\sum_{j=1}^n d_j \phi_{n_j} \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n c_i d_j \phi_{n_i} \otimes \phi_{n_j}. \end{aligned}$$

Since the $\phi_{n_i} \otimes \phi_{n_j}$ are mutually orthogonal, it follows that the matrix $(\lambda_i \delta_{i,j})$, $(i, j = 1, \dots, n)$, must equal the matrix $(c_i d_j)$, $(i, j = 1, \dots, n)$. Since the latter matrix is of rank at most one, the former is also, which means that at most one of λ_i 's is not zero. Hence $f \otimes g = 0$ or $f \otimes g = \lambda_k \phi_{n_k} \otimes \phi_{n_k}$, and in either case $f \otimes g$ is in the domain of F .

Since $\phi_1 \otimes \phi_1$ and $\phi_2 \otimes \phi_2$ are in the domain of F but $\phi_1 \otimes \phi_2$ is not, F does not have a rectangular domain.

EXAMPLE 2. We give an example of a bilinear transformation which is not completely linear. Let $\phi_1, \phi_2, \phi_3, \phi_4$ be four orthonormal elements. Let, for $\tau \neq 0$,

$$F(k(\phi_1 + \tau\phi_2), l(\phi_1 - (1/\tau)\phi_2)) = kl(\frac{1}{2}(1 + \tau)\phi_1 + \frac{1}{2}(1 - \tau)\phi_4)$$

and also

$$F(k\phi_1, l\phi_2) = kl\phi_2, \quad F(k\phi_2, l\phi_1) = kl\phi_3.$$

The proof that F is bilinear is similar to that given for Example 1.

If we take $\tau = 1$ and $\tau = -1$, then we have, respectively,

$$F(\phi_1 + \phi_2, \phi_1 - \phi_2) = \phi_1, \quad F(\phi_1 - \phi_2, \phi_1 + \phi_2) = \phi_4.$$

We also notice that

$$(\phi_1 + \phi_2) \otimes (\phi_1 - \phi_2) = (\phi_1 - \phi_2) \otimes (\phi_1 + \phi_2) + 2\phi_2 \otimes \phi_1 - 2\phi_1 \otimes \phi_2.$$

But

$$\begin{aligned} F(\phi_1 + \phi_2, \phi_1 - \phi_2) &= \phi_1 \neq \phi_4 + 2\phi_3 - 2\phi_2 \\ &= F(\phi_1 - \phi_2, \phi_1 + \phi_2) + 2F(\phi_2, \phi_1) - 2F(\phi_1, \phi_2). \end{aligned}$$

It should also be noticed, however, that the domain of F is completely

linear. For if $f \otimes g \neq 0$ is a linear combination of elements of the domain of F , one readily sees that it must be in the form

$$f \otimes g = \lambda(\phi_1 \otimes \phi_1 - \phi_2 \otimes \phi_2) + \mu\phi_1 \otimes \phi_2 + \nu\phi_2 \otimes \phi_1$$

for some constants λ , μ , and ν . As in the argument given in Example 1, this implies that the determinant $-\lambda^2 - \mu\nu = 0$. Now if λ is zero, either μ or ν is zero and $f \otimes g = \nu\phi_2 \otimes \phi_1$ or $f \otimes g = \mu\phi_1 \otimes \phi_2$; hence $f \otimes g$ is in the domain of F . If, however, λ is not zero, it may be taken as 1, and then $\mu\nu = -1$. Now if we let $\tau = \mu$, then $\nu = -1/\tau$ and

$$f \otimes g = (\phi_1 + \tau\phi_2) \otimes (\phi_1 - (1/\tau)\phi_2).$$

This also is in the domain of F .

EXAMPLE 3. We give an example of an F and \bar{F} , related as in Definition 5.2, in which F is closed with a rectangular or even a rectangular symmetric domain for which \bar{F} is nevertheless a proper extension of F .

We begin as follows. Let $\{\phi_i\}$, $(i=0, 1, 2, \dots)$, $\{\psi_j\}$, $(j=0, 1, 2, \dots)$, $\{\chi_{i,j}\}$, $(i, j=1, 2, \dots)$, be three infinite orthonormal sets of elements. Let a transformation T'' from $\mathfrak{H} \otimes \mathfrak{H}$ to \mathfrak{H} be defined by the equations

$$(1) \quad T''(\phi_0 + (1/n)\phi_n) \otimes (\psi_0 + (1/m)\psi_m) = ((n-m)^2 + 1)\chi_{n,m},$$

for $n, m=1, 2, \dots$.

Let T' be the least linear extension of T'' . (The existence of T' is easily demonstrated since the

$$(\phi_0 + (1/n)\phi_n) \otimes (\psi_0 + (1/m)\psi_m)$$

are linearly independent.) Let F' be the bilinear transformation associated with T' as in Theorem 5 above. Let F be the closure of F' .

We show that T' is closable. Let $\omega_1, \omega_2, \dots$ be a sequence in the domain of T' , such that $\omega_i \rightarrow \omega$ and $\sigma_i = T'\omega_i \rightarrow \sigma \neq 0$. Then

$$(2) \quad \sigma = \sum_{i,j} s_{i,j} \chi_{i,j},$$

where for some pair n and m , $s_{n,m} \neq 0$. Now since ω_k is in the domain of T' ,

$$(3) \quad \omega_k = \sum_{i=1}^{N_k} \sum_{j=1}^{N_k} p_{i,j}^{(k)} (\phi_0 + (1/i)\phi_i) \otimes (\psi_0 + (1/j)\psi_j)$$

and

$$\sigma_k = T'\omega_k = \sum_{i=1}^{N_k} \sum_{j=1}^{N_k} p_{i,j}^{(k)} ((i-j)^2 + 1) \chi_{i,j}.$$

Now since $\sigma = \lim_{k \rightarrow \infty} \sigma_k$,

$$s_{n,m} = (\sigma, \chi_{n,m}) = \lim_{k \rightarrow \infty} (\sigma_k, \chi_{n,m}) = \lim_{k \rightarrow \infty} p_{n,m}^{(k)}((n-m)^2 + 1).$$

Thus

$$(4) \quad \lim_{k \rightarrow \infty} p_{n,m}^{(k)} = \frac{s_{n,m}}{(n-m)^2 + 1} \neq 0.$$

On the other hand,

$$((\phi_0 + (1/i)\phi_i) \otimes (\psi_0 + (1/j)\psi_j), \phi_n \otimes \psi_m) = (1/n \cdot m) \delta_n^i \cdot \delta_m^j$$

for n and $m \geq 1$. With (3), this implies

$$(\omega_k, \phi_n \otimes \phi_m) = (1/n \cdot m) p_{n,m}^{(k)}.$$

Since $\omega = \lim_{k \rightarrow \infty} \omega_k$, (4) now implies $(\omega, \phi_n \otimes \phi_m) \neq 0$. Hence $\omega \neq 0$. Thus if $\{\omega, \sigma\}$ is a pair in the closure of the graph of T' , then $\sigma \neq 0$ implies $\omega \neq 0$. Hence closure is the graph of a transformation or $[T']$ exists.

Let \bar{F} be related to $[T']$ as in Theorem 5.

We now wish to discuss F , the closure of F' , and, in particular, its domain. To do this, we define F_0 as the bilinear transformation whose domain consists of pairs $f \otimes g$ in the form

$$f = \sum_{i=1}^{\infty} a_i(\phi_0 + (1/i)\phi_i), \quad g = \sum_{j=1}^{\infty} b_j(\psi_0 + (1/j)\psi_j),$$

and for which

$$(5) \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i|^2 \cdot |b_j|^2 ((i-j)^2 + 1)^2 < \infty,$$

and which is defined by the equation

$$(6) \quad F_0(f, g) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_i b_j ((i-j)^2 + 1) \chi_{i,j}.$$

If either f or g is zero, we simply demand that the other be in the manifold determined by the ψ 's or ϕ 's, respectively. It is easily seen that for f and g , both not zero (5) is equivalent to

$$(7) \quad \sum_{i=1}^{\infty} |a_i|^2 i^4 < \infty, \quad \sum_{j=1}^{\infty} |b_j|^2 j^4 < \infty.$$

These last conditions also insure that for f and g both not zero f shall be in the form $\sum_{i=1}^{\infty} a_i(\phi_0 + (1/i)\phi_i)$ since for this it is sufficient that $\sum_{i=1}^{\infty} a_i$ exist and that $\sum_{i=1}^{\infty} |a_i|^2/i^2 < \infty$. The second condition is trivial, and the first follows from the fact that

$$\sum_{i=1}^n |a_i| = \sum_{i=1}^n |a_i| i^2 (1/i^2) \leq \left(\sum_{i=1}^n |a_i|^2 i^4 \right)^{1/2} \left(\sum_{i=1}^n 1/i^4 \right)^{1/2}.$$

Now F_0 is closed. For suppose a sequence $f_n \otimes g_n \rightarrow f \otimes g$ for which $F_0(f_n, g_n) \rightarrow h$. If $g=0$, then $f_n \otimes g_n \rightarrow 0$, and by a proper choice of λ_n , we can let $f'_n = \lambda_n f_n$, $g = (1/\lambda_n)g_n$ in such a way that while $F_0(f'_n, g_n) = F_0(f_n, g_n) \rightarrow h$, we also have both $f_n \rightarrow 0$ and $g'_n \rightarrow 0$. Thence $a'_i(n) \rightarrow 0$, $b'_i(n) \rightarrow 0$, and (6) then implies that $(h, \chi_{i,j}) = 0$ for every i and j and hence that $h=0$. But obviously $F_0(f, 0)=0$. A similar argument will apply if $f=0$.

Now if neither f nor g is 0, then we can find sequences f'_n and g'_n such that $f'_n \rightarrow f$, $g'_n \rightarrow g$ with $F_0(f'_n, g'_n) = F_0(f_n, g_n) \rightarrow h$. These results and (6) for f'_n and g'_n yield that

$$(h, \chi_{i,j}) = a_i b_j ((i-j)^2 + 1).$$

Hence (5) holds for $f \otimes g$, and, furthermore, when we form $F_0(f, g)$ by (6), we get $F_0(f, g) = h$.

Now F_0 is obviously an extension of F' , and since it is closed it must be an extension of the closure of F' ; that is, F . But, on the other hand, if $F_0(f, g) = h$, then by taking partial sums in the expressions for f and g , given above, we see that $\{f \otimes g, h\}$ is the limit of the pairs $\{f_n \otimes g_n, h_n\}$, where $F'(f_n, g_n) = h_n$. This implies that F_0 is included in F . Thus we have $F = F_0$.

Now condition (7) above implies that the domain of F is rectangular (Definition 1.5); hence, by Theorem 6, F is completely linear. Now let $Tf \otimes g = F(f, g)$ as in Theorem 4. Furthermore $F = F_0$ is closed. If $\phi_i = \psi_i$, the domain is even symmetric.

To show our statement then, it remains only to prove that (a) F and \bar{F} are related as in Definition 5.2 and (b) \bar{F} is a proper extension of F .

(a) follows from the fact that since F is the closure of F' , $[\mathcal{U}(\mathfrak{F}')] = [\mathcal{U}(\mathfrak{F})]$ and hence $[T] = [T']$.

To prove (b), we note first that $\phi_0 \otimes \psi_0 = \bar{f}$ is not in the domain of $F = F_0$. For otherwise we would have that

$$\phi_0 = \left(\sum_{i=1}^{\infty} a_i \right) \phi_0 + \sum_{i=1}^{\infty} (a_i/i) \phi_i$$

or $a_i = 0$ for every i and $\sum_{i=1}^{\infty} a_i = 1$.

But, on the other hand, assume

$$\bar{f}_n = \sum_{i=1}^n (1/n) (\phi_0 + (1/i)\theta_i) \otimes (\psi_0 + (1/i)\psi_i) \in \mathfrak{D} \otimes \mathfrak{D}.$$

Then we obtain the relation

$$\begin{aligned}
\|\bar{f}_n - \bar{f}\|^2 &= \left\| \sum_{i=1}^n (1/n)(\phi_0 + (1/i)\phi_i) \otimes (\psi_0 + (1/i)\psi_i) - \phi_0 \otimes \psi_0 \right\|^2 \\
&= \left\| \sum_{i=1}^n ((1/n i)\phi_i \otimes \psi_0 + (1/n i)\phi_0 \otimes \psi_i + (1/i^2 n)\phi_i \otimes \psi_i) \right\|^2 \\
&= \sum_{i=1}^n (2/n^2 i^2 + 1/i^4 n^2) = (1/n^2) \left(\sum_{i=1}^n 2/i^2 + 1/i^4 \right) \\
&\rightarrow 0.
\end{aligned}$$

Also by (1) above, we obtain

$$\|T'f_n\|^2 = \left\| \sum_{i=1}^n (1/n)\chi_{i,i} \right\|^2 = \sum_{i=1}^n (1/n^2) = 1/n = 0$$

Hence $f_n \rightarrow \phi_0 \otimes \psi_0$, $T'f_n \rightarrow 0$. Thus since T' has a closure, we have $[T]\phi_0 \otimes \psi_0 = [T']\phi_0 \otimes \psi_0 = 0$. This implies that $\bar{F}(\phi_0, \psi_0)$ exists. Since $F(\phi_0, \psi_0)$ does not, we have shown (b).

EXAMPLE 4. Let $F(f, g) = f, Jg)h$, where $(,)$ denotes the inner product, J is a conjugation (cf. [14], pp. 357–365), h is some fixed element, and f and g are any two elements. Then F is easily seen to be bounded, bilinear, and with domain all $f \otimes g$. Thus F is completely linear by Theorem 6. Let T be related to F , as in Theorem 4.

But F is not hyperclosable. For if we let ϕ_1, ϕ_2, \dots be an infinite orthonormal set, then if $f_n = \sum_{i=1}^n (1/n)\phi_i \otimes J\phi_i$, $Tf_n = h$ and $f_n \rightarrow 0$. Thus T has no closed extension.

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